KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS·
Department of Mathematics and Statistics
Math 301 Final Exam (163)

Name:............................KEY...............................ID:....................................Sec:........Ser:....

Exercise/Question #	Mark	
1		20
2		20
3		20
4		20
Q1		10
Q2		10
Q3		10
Q4		10
Q5		10
Q6		10
Total		140

PART 1: Written

Exercise #1: (20 pts) Solve the boundary value problem using the Laplace transform

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \ , \ 0 < x < 1 \ , \ t > 0 \\ u(0,t) = 0 \ , \ u(1,t) = 0 \ , \ t \succeq 0 \\ u(x,0) = 0 \ , \ \frac{\partial u}{\partial t}(x,0) = 4\sin(3\pi x) \ , \ 0 < x < 1 \end{cases}$$

ANSWER:

Let $U(x,s)$ be the Laplace transform of $u(x,t)$ with respect to t. Applying the laplace transform to both sides of the PDE, we get

$$s^2 U(x,s) - su(x,0) - u_t(x,0) = c^2 \frac{d^2 U}{dx^2}$$

Thus,

$$\frac{d^2 U}{dx^2} - \left(\frac{s}{c}\right)^2 U = -4\sin(3\pi x)$$

whose general solution is

$$U = c_1 \cosh\left(\frac{s}{c}x\right) + c_2 \sinh\left(\frac{s}{c}x\right) + U_p$$

where U_p is a particular solution. Let $U_p = A\sin(3\pi x)$. Replacing into the DE we grt

$$-9\pi^2 A\sin(3\pi x) - \left(\frac{s}{c}\right)^2 A\sin(3\pi x) \equiv -\frac{4}{c^2}\sin(3\pi x)$$

Thus,

$$A = \frac{4}{s^2 + (3\pi c)^2}$$

Therefore,

$$U = c_1 \cosh\left(\frac{s}{c}x\right) + c_2 \sinh\left(\frac{s}{c}x\right) + \frac{4}{s^2 + (3\pi c)^2}\sin(3\pi x)$$

Since $U(0,s) = 0$ and $U(1,s) = 0$, we obtain $c_1 = c_2 = 0$, giving,

$$U(x,s) = \frac{4}{s^2 + (3\pi c)^2}\sin(3\pi x)$$

whose inverse Laplace transform is,

$$u(x,t) = \frac{4}{3\pi c}\sin(3\pi ct)\sin(3\pi x)$$

Exercise #2: (20 pts) Solve the following problem using the Fourier transform leaving the solution in integral form

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \ , \ -\infty < x < \infty \ , \ t > 0 \\ u(x,0) = e^{-3|x|}, \end{cases}$$

where $\lim_{|x|\to\infty} u(x,t) = 0$ and $\lim_{|x|\to\infty} u_x(x,t) = 0$.

ANSWER:

Let $U(\alpha,t)$ be the Fourier transform of $u(x,t)$ with respect to x. Applying the Fourier transform to both sides of the PDE, we get

$$\frac{dU}{dt} = k(-i\alpha)^2 U$$

We have

$$\begin{aligned} U(\alpha,0) &= \int_{-\infty}^{\infty} u(x,0)e^{i\alpha x}dx = \int_{-\infty}^{\infty} e^{-3|x|}e^{i\alpha x}dx = \int_{-\infty}^{0} e^{3x}e^{i\alpha x}dx + \int_{-0}^{\infty} e^{-3x}e^{i\alpha x}dx \\ &= \lim_{a\to-\infty} \frac{1-e^{(3+i\alpha)x}}{3+i\alpha} + \lim_{b\to\infty} \frac{e^{(-3+i\alpha)x}-1}{-3+i\alpha} = \frac{1}{3+i\alpha} + \frac{1}{3-i\alpha} = \frac{6}{\alpha^2+9} \end{aligned}$$

thus,

$$U(\alpha,t) = \frac{6}{\alpha^2+9}e^{-k\alpha^2 t}$$

so that the solution is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{6}{\alpha^2+9}e^{-k\alpha^2 t}e^{-i\alpha x}d\alpha$$

which can be rewritten as

$$u(x,t) = \frac{6}{\pi}\int_{0}^{\infty} \frac{1}{\alpha^2+9}e^{-k\alpha^2 t}\cos(\alpha x)d\alpha$$

Exercise #3: (20 pts) Consider the Sturm-Liouville problem

$$\begin{cases} y'' + y' = \lambda y \quad . \quad , & 0 < x < 1 \\ y(0) = 0 \quad , & y(1) = 0 \end{cases}$$

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(a) Write the differential equation in self-adjoint form specifying the weight function as well as an orthogonality relation.
(b) Find the eigenvalues and corresponding eigenfunctions
(c) Use (a) and (b) above to obtain the eigenfunctions expansion of $f(x) = e^{-\frac{x}{2}}$, $0 < x < 1$ as well as the value of the series at $x = \frac{1}{2}$.
ANSWER:
(a) An integrating factor is e^x so that a self-adjoint form is

$$-\left(e^x y'\right)' = (-\lambda) e^x y$$

from which we get the weight function $w(x) = e^x$. In view of the boundary conditions we conclude it is a regular Sturm-Liouville problem. Thus, and orthogonality condition for the eigenfunctions y_k is

$$\int_0^1 w(x) y_k(x) y_l(x) dx = 0 \text{ if } k \neq l$$

and an associated inner product is

$$< f, g >_w = \int_0^1 w(x) f(x) g(x) dx$$

(b) Eigenvalues and eigenfunctions:
The differential equation $y'' + y' - \lambda y = 0$ is a linear constant coefficients whose characteristic equation is $r^2 + r - \lambda = 0$. The discriminant is $\Delta = 1 + 4\lambda$. We distinguish three cases, $\Delta = 0$. $\Delta > 0$ and $\Delta < 0$.

Case $\Delta = 0$. It leads to one double root $r = -\frac{1}{2}$ so that the general solution is $y = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}}$.Using the boundary conditions we get $y = 0$ (trivial solution).

Case $\Delta > 0$ We let $\Delta = \varpi^2 > 0$ so that we have two real roots $r_1 = \frac{-1-\varpi}{2}$ and $r_2 = \frac{-1+\varpi}{2}$. Hence the general solution is $y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$. Using the boundary conditions we get again the trivial solution $y = 0$.

Case $\Delta < 0$ We let $\Delta = -\varpi^2 < 0$ so that we have two complex conjugate roots $r_1 = \frac{-1-i\varpi}{2}$ and $r_2 = \frac{-1+i\varpi}{2}$. Hence the general solution is $y = e^{-\frac{x}{2}}\left[c_1 \cos(\frac{\varpi}{2}x) + c_2 \sin(\frac{\varpi}{2}x)\right]$. Using the boundary conditions we get this time, $c_1 = 0$ and $c_2 \sin(\frac{\varpi}{2}) = 0$, Since c_2 cannot be zero (trivial solution), $\sin(\frac{\varpi}{2}) = 0$. Thus, $\varpi_k = 2k\pi.$, $k \geq 1$ leading to the eigenvalues $\lambda_k = -\frac{1}{4} - (k\pi)^2$, $k \geq 1$. with eigenfunctions $y_k = e^{-\frac{x}{2}} \sin(k\pi x)$, $k \geq 1$.

(c) An eigenfunctions expansion for f is $f(x) \sim \sum_{k \geq 1} c_k y_k(x)$ where

$$\begin{aligned} c_k &= \frac{\int_0^1 e^x f(x) y_k(x) dx}{\int_0^1 e^x \left|y_k(x)\right|^2 dx} = \frac{\int_0^1 e^x e^{-\frac{x}{2}} e^{-\frac{x}{2}} \sin(k\pi x) dx}{\int_0^1 e^x \left|e^{-\frac{x}{2}} \sin(k\pi x)\right|^2 dx} = \frac{\int_0^1 \sin(k\pi x) dx}{\int_0^1 \sin^2(k\pi x) dx} \\ &= 2\int_0^1 \sin(k\pi x) dx = \frac{2}{k\pi}\left(1 - \cos(k\pi)\right) = \frac{2}{k\pi}\left[1 - (-1)^k\right] \\ &= \frac{4}{k\pi} \text{ if } k \text{ is odd and zero if } k \text{ even.} \end{aligned}$$

so letting $k = 2n + 1$ leads to

$$f(x) \sim \sum_{n \geq 0} \frac{4}{(2n+1)\pi} y_{2n+1}(x)$$

Since $x = \frac{1}{2}$ is a continuity point for the function, the value of the series at $x = \frac{1}{2}$ is $f(\frac{1}{2}) = e^{-\frac{1}{4}}$

Exercise #4: (20 pts)
(a) (10 pts) Find the first four terms of the Fourier Legendre series for the function

$$f(x) = \begin{cases} -1\,, & -1 < x < 0 \\ 2\,, & 0 < x < 1 \end{cases}$$

ANSWER:
The Fourier Legendre series of f is

$$f(x) \sim \sum_{n\geq 9} c_n P_n(x)$$

where $c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$, $n \geq 0$, $P_n(x)$ being the the Legendre polynomials. We have

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{1}{2}(3x^2-1),\ P_3(x) = \frac{1}{2}(5x^3-3x)$$

thus,

$$\begin{aligned}
c_0 &= \frac{1}{2}\int_{-1}^{1} f(x)P_0(x)dx = -\frac{1}{2}\int_{-1}^{0} dx + \frac{1}{2}\int_{0}^{1} 2dx = -\frac{1}{2} + 1 = \frac{1}{2} \\
c_1 &= \frac{3}{2}\int_{-1}^{1} f(x)P_1(x)dx = -\frac{3}{2}\int_{-1}^{0} xdx + \frac{3}{2}\int_{0}^{1} 2xdx = --\frac{3}{4} + \frac{3}{2} = \frac{9}{4} \\
c_2 &= \frac{5}{2}\int_{-1}^{1} f(x)P_2(x)dx = -\frac{5}{4}\int_{-1}^{0} (3x^2-1)dx + \frac{5}{2}\int_{0}^{1} (3x^2-1)dx = 0 \\
c_3 &= \frac{7}{2}\int_{-1}^{1} f(x)P_3(x)dx = -\frac{7}{4}\int_{-1}^{0} (5x^3-3x)dx + \frac{7}{2}\int_{-0}^{1} (5x^3-3x)dx = \frac{7}{4}(\frac{5}{4}-\frac{3}{2}) + \frac{7}{2}(\frac{5}{4}-\frac{3}{2}) \\
&= -\frac{7}{16} - \frac{7}{8} = -\frac{21}{16}
\end{aligned}$$

Therefore the Fourier Bessel series is

$$f(x) \sim \frac{1}{2}P_0(x) + \frac{9}{4}P_1(x) + 0P_2(x) - \frac{21}{16}P_3(x)$$

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.**(b)** (10 pts) Find the Fourier integral representation of the function

$$f(x) = \begin{cases} 0\,, & x < -2 \\ -1\,, & -2 < x < 0 \\ -2\,, & 0 < x < 2 \\ 0\,, & x > 2 \end{cases}$$

ANSWER:
The Fourier integral representation of f is

$$f(x) \sim \frac{1}{\pi}\int_{0}^{\infty} \left[A(\alpha)\cos(\alpha x) + B(\alpha)\sin(\alpha x)\right] dx$$

where

$$
\begin{aligned}
A(\alpha) &= \int_{-\infty}^{\infty} f(x)\cos(\alpha x)dx = \int_{-2}^{0} \cos(\alpha x)dx - 2\int_{-0}^{2} \cos(\alpha x)dx \\
&= -\frac{\sin(2\alpha)}{\alpha} - 2\frac{\sin(2\alpha)}{\alpha} = -3\frac{\sin(2\alpha)}{\alpha} \\
B(\alpha) &= \int_{-\infty}^{\infty} f(x)\sin(\alpha x)dx = -\int_{-2}^{0} \sin(\alpha x)dx - \int_{0}^{2} 2\sin(\alpha x)dx \\
&= \frac{1-\cos(2\alpha)}{\alpha} + 2\frac{\cos(2\alpha)-1}{\alpha} = \frac{1}{\alpha}\left(\cos(2\alpha)-1\right)
\end{aligned}
$$

Hence

$$
f(x) \sim \frac{1}{\pi}\int_{0}^{\infty} \left[-3\frac{\sin(2\alpha)}{\alpha}\cos(\alpha x) + \frac{1}{\alpha}\left(\cos(2\alpha)-1\right)\sin(\alpha x)\right] d\alpha
$$

Part 2: MCQ

Q1) (10 pts) Consider the equation $y(t) = \cos t + \int_0^t e^{-\tau} y(t-\tau) d\tau$ and let $Y(s)$ denote the Laplace transform of $y(t)$ then

(A) $Y(s) = \frac{s^2-1}{s^2+1}$

(B) $Y(s) = \frac{1}{s^2+1}$

(C) $Y(s) = \frac{s+2}{s^2-1}$

(D) $Y(s) = \frac{s+1}{s^2+1}$

(E) $Y(s) = \frac{s+2}{s^2+1}$

ANSWER:

Applying The Laplace transform to both sides of the integral equation and using the properties of the Laplace transform, we get,

$$Y(s) = \frac{s}{s^2+1} + \frac{1}{s+1} Y(s)$$

so that

$$Y(s) = \frac{s+1}{s^2+1}$$

so (D).

Q2) (10 pts) The inverse Laplace transform of

$$F(s) = \frac{2s^2 - s + 1}{s^2 + s + 1}$$

is

(A) $2\delta(t) - 3e^{-\frac{1}{2}t}\left[\cos\left(\frac{1}{2}t\right) - \frac{1}{9}\sqrt{3}\sin\left(\frac{1}{2}t\right)\right]$

(B) $2u(t) - 3e^{-\frac{1}{2}t}\left[\cos\left(\sqrt{3}t\right) - \frac{1}{9}\sqrt{3}\sin\left(\sqrt{3}t\right)\right]$

(C) $\delta(t) - 3e^{-t}\left[\cos\left(\frac{1}{2}\sqrt{3}t\right) - \frac{1}{9}\sqrt{3}\sin\left(\frac{1}{2}\sqrt{3}t\right)\right]$

(D) $2\delta(t) - 2e^{-\frac{1}{2}t}\left[\cos\left(\sqrt{3}t\right) - \frac{1}{9}\sqrt{3}\sin\left(\sqrt{3}t\right)\right]$

(E) $2\delta(t) - 3e^{-\frac{1}{2}t}\left[\cos\left(\frac{1}{2}\sqrt{3}t\right) - \frac{1}{3\sqrt{3}}\sin\left(\frac{1}{2}\sqrt{3}t\right)\right]$

ANSWER:

We have,

$$\begin{aligned} F(s) &= 2 - \frac{3s+1}{s^2+s+1} = 2 - \frac{3\left(s+\frac{1}{2}\right) - \frac{1}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= 2 - \frac{3\left(s+\frac{1}{2}\right)}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \end{aligned}$$

whose inverse Laplace transform is

$$f(t) = 2\delta(t) - 3e^{-\frac{t}{2}}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{1}{3\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

so (E)

Q3) (10 pts) The Laplace transform of $f(t) = te^{-2t}\sin(3t)$, .is

(A) $\frac{6s+12}{(s^2+4s+13)^2}$
(B) $\frac{6s+1}{(s^2+2s+13)^2}$
(C) $\frac{s+12}{(s^2+4s+1)^2}$
(D) $\frac{6s+12}{(s^2+4s+9)^2}$
(E) $\frac{6s-12}{(s^2-4s+13)^2}$

ANSWER:

Let $g(t) = e^{-2t}\sin(3t)$. Its Laplace transform is $G(s) = \frac{3}{(s+2)^2+9}$. So the Laplace transform of $f(t)$ is

$$F(s) = -\frac{d}{ds}\left[\frac{3}{(s+2)^2+9}\right] = \frac{3(2s+4)}{[(s+2)^2+9]^2} = \frac{6s+12}{[s^2+4s+13]^2}$$

thus (A)

Q4) (10 pts) The Fourier series of $f(x) = x$, $-\pi < x < \pi$ is

(A) $\sum_{k\succeq 1} \frac{2(-1)^{k+1}}{\pi}\sin(kx)$
(B) $\sum_{k\succeq 1} \frac{(-1)^{k+1}}{\pi}\sin(kx)$
(c) $\sum_{k\succeq 1} \frac{(-1)^{k+1}}{\pi^2}\sin(kx)$
(D) $\sum_{k\succeq 1} \frac{2(-1)^{k+1}}{\pi}\cos(kx)$
(E) $\sum_{k\succeq 1} \frac{2(-1)^{k+1}}{k}\sin(kx)$

ANSWER:

f is odd. so its Fourier series is

$$f(x) = \sum_{k=1}^{\infty} c_k \sin\left(\frac{k\pi x}{\pi}\right) = \sum_{k=1}^{\infty} c_k \sin(kx)$$

where

$$\begin{aligned} c_k &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)dx = \frac{2}{\pi}\int_0^{\pi} x\sin(kx)dx \\ &= \frac{2}{\pi}\left[-x\frac{\cos(kx)}{k}|_{x=0}^{x=\pi} + \int_0^{\pi}\frac{\cos(kx)}{k}dx\right] \\ &= -\frac{2}{k}\cos(k\pi) = \frac{2(-1)^{k+1}}{k} \end{aligned}$$

so that the Fourier series is

$$\sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k}\sin(kx)$$

thus (E)

Q5) (10 pts) The Fourier-Bessel series expansion of $f(x) = x$, $0 < x < 3$, in Bessel functions of order one that satisfy the boundary condition $J_1(3\alpha) = 0$ is

(A) $2\sum_{n\geq 1} \frac{1}{\alpha_n J_2(2\alpha_n)} J_1(3\alpha_n x)$
(B) $\sum_{n\geq 1} \frac{1}{\alpha_n J_2(3\alpha_n)} J_1(\alpha_n x)$
(C) $2\sum_{n\geq 1} \frac{1}{\alpha_n J_2(\alpha_n)} J_1(\alpha_n x)$
(D) $3\sum_{n\geq 1} \frac{(-11)^n}{\alpha_n J_2(3\alpha_n)} J_1(\alpha_n x)$
(E) $2\sum_{n\geq 1} \frac{1}{\alpha_n J_2(3\alpha_n)} J_1(\alpha_n x)$
ANSWER:

Let $\alpha_n, n \geq 1$, be the solutions of $J_1(3\alpha) = 0$.The Fourier-Bessel series expansion of f is

$$f(x) \sim \sum_{n\geq 1} c_n J_1(\alpha_n x)$$

where

$$\begin{aligned} c_n &= \frac{2}{3^2 J_2^2(3\alpha_n)} \int_0^3 x J_1(\alpha_n x) f(x) dx \\ &= \frac{2}{9 J_2^2(3\alpha_n)} \int_0^3 x^2 J_1(\alpha_n x) dx \end{aligned}$$

But

$$\frac{d}{dy}\left[y^2 J_2(y)\right] = y^2 J_1(y)$$

so let $y = \alpha_n x$, we have $dy = \alpha_n dx$ and c_n becomes

$$\begin{aligned} c_n &= \frac{2}{9\alpha_n^3 J_2^2(3\alpha_n)} \int_0^{3\alpha_n} y^2 J_1(y) dy \\ &= \frac{2}{9\alpha_n^3 J_2^2(3\alpha_n)} \int_0^{3\alpha_n} \frac{d}{dy}\left[y^2 J_2(y)\right] dy \\ &= \frac{2}{\alpha_n J_2(3\alpha_n)} \end{aligned}$$

so the Fourier Bessel series of f is

$$f(x) \sim \sum_{n\geq 1} \frac{2}{\alpha_n J_2(3\alpha_n)} J_1(\alpha_n x)$$

Thus (E).
Q6) (10 pts) The general solution of $x^2 y'' + xy' + (3x^2 - 1)y = 0$, is:

(A) $c_1 J_1(x) + c_2 Y_1(x)$
(B) $c_1 J_1\left(\sqrt{3}x\right) + c_2 Y_1\left(\sqrt{3}x\right)$
(C) $c_1 J_{\sqrt{3}}(x) + c_2 Y_{\sqrt{3}}(x)$
(D) $c_1 J_1\left(\sqrt{3}x\right) + c_2 J_{-1}\left(\sqrt{3}x\right)$
(E) $c_1 J_0\left(\sqrt{3}x\right) + c_2 Y_0\left(\sqrt{3}x\right)$
ANSWER:

This is a parametric Bessel equation $x^2 y'' + xy' + (\alpha^2 x^2 - \nu^2)y = 0$ with $\alpha = \sqrt{3}$ and $\nu = 1$. Therefore, its general solution is

$$y = c_1 J_1(\sqrt{3}x) + c_2 Y_1(\sqrt{3}x)$$

thus (B).