King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

Math 202-Final Exam- Term 163
Duration 180 minutes

Name: Key ID#:

Section Number: Serial Number:

Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly
3. Show all your work. No points for answers without justification.
4. Make sure you have 16 pages of problems (Total of 16 Problems)
5. DE means Differential equation.

Question Number	Points	Maximum Points
1		8
2		8
3		8
4		8
5		8
6		8
7		8
8		8
9		12
10		12
11		8
12		12
13		8
14		8
15		8
16		8
Total		140

1. Consider the DE

$$(y + y^2 \cos x)\, dx + (2x + 3y \sin x)\, dy = 0,\ y > 0.$$

(a) **[4 points]** Show that the DE is not exact.

$M(x,y) = y + y^2 \cos x$

$N(x,y) = 2x + 3y \sin x$

$M_y = 1 + 2y \cos x$ (1 pt)

$N_x = 2 + 3y \cos x$ (1pt)

Since $M_y \neq N_x$ (2pts)

$\therefore$ The DE is Not exact.

(b) **[4 points]** Find an integrating factor that transforms into an exact DE.

(Do not solve the new DE)

$$\frac{N_x - M_y}{M} = \frac{1 + y\cos x}{y(1 + y\cos x)} \quad (1pt)$$

$$= \frac{1}{y} \quad (1pt)$$

$\therefore$ an integrating factor is

$$\mu(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y. \quad (2pts)$$

2. **[8 points]** Find a family of solutions for the given homogeneous DE

$$(y + \sqrt{xy})\,dx - x\,dy = 0, \quad x > 0, y > 0.$$

(1pt) (1pt)

let $y = ux \Rightarrow dy = u\,dx + x\,du$

Substituting in the DE, we get

$(ux + \sqrt{ux^2})\,dx - x\,(u\,dx + x\,du) = 0$ (1 pt)

$\Rightarrow x\,(u + \sqrt{u})\,dx - xu\,dx - x^2\,du = 0$

$\Rightarrow \cancel{u\,dx} + \sqrt{u}\,dx - \cancel{u\,dx} - x\,du = 0$ (1pt)

$\Rightarrow \dfrac{dx}{x} - \dfrac{du}{\sqrt{u}} = 0$ (1pt)

$\Rightarrow \ln x - 2\sqrt{u} = C$ (2pts)

$\Rightarrow \ln x - 2\sqrt{\dfrac{y}{x}} = C$ (1pt)

3. **[8 points]** The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 5 hours 1600 bacteria are present. What was the initial number of bacteria?

Let $X(t)$ be the number of bacteria present at time t.

$$\frac{dx}{dt} = k\cancel{X}(t) \quad \text{(2pts)}, \qquad X(3) = 400$$

$$X(5) = 1600$$

$$X(0) = ?$$

$$X(t) = Ce^{kt} \quad \text{(2pts)}$$

$$X(3) = 400 \Rightarrow Ce^{3k} = 400 \quad \ldots (1)$$

$$X(5) = 1600 \Rightarrow Ce^{5k} = 1600 \quad \ldots (2)$$

$$\frac{Eq(2)}{Eq(1)} \Rightarrow e^{2k} = 4 \Rightarrow 2k = \ln 4$$

$$\Rightarrow k = \ln 2 \quad \text{(2pts)}$$

From Eq(1), we get

$$Ce^{3\ln 2} = 400 \Rightarrow 8C = 400$$

$$\Rightarrow C = 50 \quad \text{(1pt)}$$

$$\therefore X(t) = 50(2)^t$$

$$X(0) = 50 \,. \quad \text{(1pt)}$$

4. **[8 points]** Let L be a linear differential operator such that y_1 and y_2 are respectively, particular solutions of the differential equations

$$L(y) = 3\cos(2x) \text{ and } L(y) = (x+1)^2.$$

Use the superposition principle to find a particular solution of

$$L(y) = 2\cos^2 x + x^2 + 2x.$$

(Write your answer in terms of y_1 and y_2).

Let $g_1(x) = 3\cos(2x)$, $g_2(x) = (x-1)^2$, $g_3(x) = 2\cos^2 x + x^2 + 2x$

Given $L(y_1) = g_1(x)$ and $L(y_2) = g_2(x)$

Note that

$$g_3(x) = 2\cos^2 x + x^2 + 2x$$

$$= \cancel{2}\left(\frac{1+\cos(2x)}{\cancel{2}}\right) + x^2 + 2x$$

$$= \cos(2x) + (x+1)^2$$

$$= \underset{(2pts)}{\frac{1}{3} g_1(x)} + \underset{(2pts)}{g_2(x)}$$

$\therefore$ by superposition principle

$$y = \overset{(2pts)}{\frac{1}{3} y_1} + \overset{(2pts)}{y_2} \text{ is a particular solution}$$

of $L(y) = g_3(x)$

5. **[8 points]** Solve

$$D^2(D-1)(D+2)^2(D^2+16)(D^2-6D+13)^2y=0.$$

The auxiliary equation is

$$m^2(m-1)(m+2)^2(m^2+16)(m^2-6m+13)^2=0$$

$$\Rightarrow m=0,0,1,-2,-2,\pm 4i,3\pm 2i,3\pm 2i$$

(2 pts)

The general solution is

$$y=\overset{1pt}{c_1+c_2x}+\overset{(1pt)}{c_3e^{x}+(c_4+c_5x)e^{-2x}}+\overset{(1pt)}{c_6\cos(4x)+c_7\sin(4x)}$$

$$+\underset{(1pt)}{(c_8+c_9x)e^{3x}\cos(2x)}+\underset{(1pt)}{(c_{10}+c_{11}x)e^{3x}\sin(2x)}.$$

6. **[8 points]** Use the undetermined coefficients-annihilator approach method to find the most suitable form of a particular solution for the DE:

$$D^3(D^2+D+1)\,y = x^2 + x\,e^{3x} \qquad \cdots (1)$$

(Do not evaluate the constants)

$$D^3(D^2+D+1)\,y = 0 \qquad \cdots (2)$$

The auxiliary equation is $m^3(m^2+m+1)=0$

$\Rightarrow m = 0$ (order 3), $m = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

(2 pts) $y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + c_5 e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)$

An annihilator $A(D)$ of $x^2 + x e^{3x}$ is

(2 pts) $A(D) = D^3 \cdot (D-3)^2$.

Operating $A(D)$ to both sides of (1) yields

(1 pt) $D^6 (D-3)^2 (D^2+D+1)\, y = 0 \qquad \cdots (3)$

The auxiliary equation of (3) has the roots:

(1 pt) 0 (order 6), 3 (order 2), $-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

The general solution of (3) is

(1 pt) $y = b_1 + b_2 x + b_3 x^2 + \overbrace{b_4 x^3 + b_5 x^4 + b_6 x^5 + (b_7 + b_8 x)\, e^{3x}}^{y_p}$

y_c $+\; b_9 e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) + b_{10} e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)$

$\therefore\; y_p = A x^5 + B x^4 + C x^3 + E x e^{3x} + F e^{3x}$ (1 pt)

7. **[8 points]** Find a particular solution of

$$xy'' + 2xy' + xy = e^{-x},\ x > 0$$

Putting the equation in standard form gives

$$y'' + 2y' + y = \frac{e^{-x}}{x}$$

The auxiliary equation is $m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$. Thus

(1pt)

$y_c = c_1 e^{-x} + c_2 x e^{-x}$. Let $y_p = u_1(x) e^{-x} + u_2(x) x e^{-x}$

$$W = \begin{vmatrix} e^{-x} & x e^{-x} \\ -e^{-x} & e^{-x} - x e^{-x} \end{vmatrix} = e^{-2x} \quad \text{(1pt)}$$

$$W_1 = \begin{vmatrix} 0 & x e^{-x} \\ \frac{e^{-x}}{x} & e^{-x} - x e^{-x} \end{vmatrix} = -e^{-2x} \quad \text{(1pt)}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \frac{e^{-x}}{x} \end{vmatrix} = \frac{e^{-2x}}{x} \quad \text{(1pt)}$$

$$u_1(x) = \int \frac{W_1}{W}\,dx = \int -1\,dx = -x$$

$$u_2(x) = \int \frac{W_2}{W}\,dx = \int \frac{1}{x}\,dx = \ln x$$

(3pts)

$\therefore\ y_p = -x e^{-x} + x e^{-x} \ln x$ or $y_p = x e^{-x} \ln x$.

(1pt)

8. **[8 points]** Determine the singular points of

$$x(x^2-4)^2 y'' + 3(x+2)y' + (x-2)\, y = 0.$$

Classify each singular point as regular or irregular.(**Justify your answer.**)

Putting the equation in standard form gives

$$y'' + \frac{3}{x(x-2)^2(x+2)}\, y' + \frac{1}{x(x+2)^2(x-2)}\, y = 0.$$

$$P(x) = \frac{3}{x(x-2)^2(x+2)} \quad (1\text{pt}), \qquad Q(x) = \frac{1}{x(x+2)^2(x-2)} \quad (1\text{pt})$$

The singular points are : $x=0$, $x=2$ and $x=-2$. (3 pts)

(1pt) $x=0$ is regular singular point since

$$x\,P(x) = \frac{3}{(x-2)^2(x+2)} \quad \text{and}$$

$$x^2 Q(x) = \frac{x}{(x+2)^2(x-2)} \quad \text{are analytic at } x=0.$$

(1pt) $x=-2$ is also a regular singular point since

$$(x+2)\,P(x) = \frac{3}{x(x-2)^2},$$

$$(x+2)^2 Q(x) = \frac{1}{x(x-2)} \quad \text{are analytic at } x=-2.$$

(1pt) $x=2$ is irregular singular point since

$$(x-2)\,P(x) = \frac{3}{x(x-2)(x+2)} \quad \text{is Not analytic at } x=2.$$

9. **[12 points]** Use power series about the ordinary point $x = 0$ to solve

$$(x^2 - 1)y'' - 2y = 0.$$

(If the series solution is infinite write only the first three nonzero terms).

let $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.

(3 pts)

Substituting in the DE gives

(1 pt) $\sum_{n=2}^{\infty} n(n-1) a_n x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} 2 a_n x^n = 0$

$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^n - 2a_2 - 6a_3 x - 2a_0 - 2a_1 x - \sum_{n=4}^{\infty} n(n-1) a_n x^{n-2}$ (1 pt)

$- \sum_{n=2}^{\infty} 2 a_n x^n = 0$

$\Rightarrow -2(a_2 + a_0) - 2(3a_3 + a_1) x + \sum_{n=2}^{\infty} (n^2 - n - 2) a_n x^n - \sum_{n=2}^{\infty} (n+2)(n+1) a_{n+2} x^n = 0$

(1 pt) (1 pt) (2 pts)

$\Rightarrow a_2 = -a_0, \quad a_3 = -\frac{1}{3} a_1, \quad a_{n+2} = \frac{n^2 - n - 2}{(n+2)(n+1)} a_n, \quad n = 2, 3, \ldots$

$\Rightarrow a_4 = 0, \quad a_5 = \frac{1}{5} a_3 = \frac{-1}{3 \cdot 5} a_1, \quad a_6 = 0,$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cancel{a_4 x^4}^{\,0} + a_5 x^5 + \ldots$

$= a_0 (1 - x^2) + a_1 \left(x - \frac{1}{3} x^3 - \frac{1}{15} x^5 + \ldots\right), \quad |x| < 1$

(1 pt) (2 pts)

10. **[12 points]** Find the first three nonzero terms of the series solution of the DE $4xy'' + 2y' + y = 0$ about $x = 0$ corresponding to the largest indicial root.

let $y = \sum_{n=0}^{\infty} a_n x^{n+r} \Rightarrow y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$

$\Rightarrow y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2}$. (3pts)

Substituting in the DE gives

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1) a_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r) a_n x^{n+r-1} + \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow x^r \left[\left(4r(r-1) + 2r\right) a_0 x^{-1} + \sum_{n=1}^{\infty} 2(n+r)(2n+2r-1) a_n x^{n-1} + \sum_{n=1}^{\infty} a_{n-1} x^{n-1} \right] = 0$$

(1pt)

let $a_0 \neq 0$

$\Rightarrow 2r(2r-1) = 0$ (2pts), $\quad a_n = \dfrac{-1}{2(n+r)(2n+2r-1)} a_{n-1}, \ n = 1, 2, \ldots$ (2pts)

$\Rightarrow r = 0, \ r = \frac{1}{2}$ (1pt)

for $r = \frac{1}{2} \Rightarrow a_n = \dfrac{-1}{2n(2n+1)} a_{n-1}, \quad n = 1, 2, \ldots$ (1pt)

$a_1 = \dfrac{-1}{6} a_0, \quad a_2 = \dfrac{-1}{20} \cdot a_1 = \dfrac{1}{120} a_0$

$\therefore$ The solution corresponding to $r = \frac{1}{2}$ is (Take $a_0 = 1$)

$$y(x) = x^{\frac{1}{2}}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 + \ldots\right)$$

(2pts)

11. **[8 points]** Without solving the system verify that

$$X_1 = \begin{pmatrix} -e^{-t} \\ e^{-t} \end{pmatrix} \text{ and } X_2 = \begin{pmatrix} e^{5t} \\ 2e^{5t} \end{pmatrix}$$

form a fundamental set of solutions of the system $X' = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} X$ on the interval $(-\infty, \infty)$.

It is enough to show that $\phi(t) = \begin{pmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{pmatrix}$ is a fundamental matrix for the system.

1) We show that $\phi(t)$ satisfies the system

(4 pts)

$$\text{L.H.S} = \phi'(t) = \begin{pmatrix} e^{-t} & 5e^{5t} \\ -e^{-t} & 10e^{5t} \end{pmatrix}$$

$$\text{R.H.S} = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{pmatrix} = \begin{pmatrix} e^{-t} & 5e^{5t} \\ -e^{-t} & 10e^{5t} \end{pmatrix}$$

$\therefore$ $\phi(t)$ satisfies the system.

2) We show that X_1 and X_2 are L.I or $\det\phi(t) \neq 0$.

(4pts) $$\begin{vmatrix} -e^{-t} & e^{5t} \\ e^{-t} & 2e^{5t} \end{vmatrix} = -2e^{4t} - e^{4t} = -3e^{4t} \neq 0$$

for all $t \in (-\infty, \infty)$

From 1) and 2), X_1 and X_2 form a fundamental set of solution of the given system.

12. **[12 points]** Find the general solution of the system $X' = AX$ where

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & -1 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} -\lambda & -1 \\ -1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(\lambda^2-1) = 0 \Rightarrow \lambda = 2, 1, -1$. (3pts)

For $\lambda = 2$: let $K_{\lambda=2} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$ be a soln. of

$$\left(\begin{array}{ccc|c} -2 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right) \Rightarrow k_1 = -2k_3, \quad -2k_1 - k_2 - k_3 = 0$$

let $k_3 = 1 \Rightarrow k_1 = -2 \Rightarrow k_2 = 3$.

$\therefore K_{\lambda=2} = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$. (2pts)

For $\lambda = 1$: $\left(\begin{array}{ccc|c} -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right) \Rightarrow k_2 = 0, \ k_1 = -k_3$

let $k_3 = 1 \Rightarrow k_1 = -1$ and $K_{\lambda=1} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. (2pts)

For $\lambda = -1$: $\left(\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right) \Rightarrow k_2 = 0, \ k_1 = k_3$

$\therefore K_{\lambda=-1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (2pts). So the general solution is

$$X = c_1 \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} e^{t} + c_3 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-t} \quad (3\text{pts})$$

13. **[8 points]** Given that $\lambda = 4 + i$ is an eigenvalue of the matrix $A = \begin{pmatrix} 11 & -10 \\ 5 & -3 \end{pmatrix}$, find the general solution of the homogeneous system $X' = AX$.

$$|A - \lambda I| = \begin{vmatrix} 11-\lambda & -10 \\ 5 & -3-\lambda \end{vmatrix}$$

To find the eigenvector $K = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ corresponding to $\lambda = 4+i$, we solve

$$\left(\begin{array}{cc|c} 7-i & -10 & 0 \\ 5 & -7-i & 0 \end{array}\right) \text{(1 pt)} \Rightarrow k_2 = \frac{7-i}{10} k_1$$

$$K_{\lambda=4+i} = \begin{pmatrix} 10 \\ 7-i \end{pmatrix} = \begin{pmatrix} 10 \\ 7 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} i \quad \text{(2 pts)}$$

$$X_1 = e^{4t}\left[\begin{pmatrix} 10 \\ 7 \end{pmatrix} \cos t - \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin t \right] \quad \text{(2 pts)}$$

$$X_2 = e^{4t}\left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} \cos t + \begin{pmatrix} 10 \\ 7 \end{pmatrix} \sin t \right] \quad \text{(2 pts)}$$

The general solution is

$$X = c_1 X_1 + c_2 X_2 . \quad \text{(1 pt)}$$

14. **[8 points]** If $X_1 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{2t}$ is a solution of the system

$$X' = \begin{pmatrix} -4 & 9 \\ -4 & 8 \end{pmatrix} X$$

corresponding to the eigenvalue $\lambda = 2$ of multiplicity 2, then find a second solution X_2 such that X_1 and X_2 are linearly independent.

$K_{\lambda=2} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$. We need to find $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$ s.t

(2 pts)

$$\begin{pmatrix} -4-\lambda & 9 \\ -4 & 8-\lambda \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$$

That is;

$$\left(\begin{array}{cc|c} -6 & 9 & 3 \\ -4 & 6 & 2 \end{array}\right) \xrightarrow[-\frac{1}{2}R_2]{-\frac{1}{3}R_1} \left(\begin{array}{cc|c} 2 & -3 & -1 \\ 2 & -3 & -1 \end{array}\right)$$

$\Rightarrow 2P_1 - 3P_2 = -1$. Take $P_2 = 0 \Rightarrow P_1 = -\frac{1}{2}$

$\therefore P = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$. (2 pts)

$$X_2 = (Kt + P)\, e^{\lambda t} \quad (3 \text{ pts})$$

$$= \begin{pmatrix} 3t - \frac{1}{2} \\ 2t \end{pmatrix} e^{2t}. \quad (1 \text{ pt})$$

15. **[8 points]** Use variation of parameters to find a particular solution of $X' = AX + \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ given that $X_c = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} e^{5t}$ is the general solution of $X' = AX$.

$$\Phi(t) = \begin{pmatrix} 0 & 3e^{5t} \\ e^{-t} & 2e^{5t} \end{pmatrix} \quad (1\,\text{pt})$$

$$\Phi^{-1}(t) = \frac{1}{-3e^{4t}} \begin{pmatrix} 2e^{5t} & -3e^{5t} \\ -e^{-t} & 0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3}e^{t} & e^{t} \\ \frac{1}{3}e^{-5t} & 0 \end{pmatrix} \quad (2\,\text{pts})$$

$$\Phi^{-1}(t)F(t) = \begin{pmatrix} -\frac{2}{3}e^{t} & e^{t} \\ \frac{1}{3}e^{-5t} & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{7}{3}e^{t} \\ \frac{5}{3}e^{-5t} \end{pmatrix}$$

$$\therefore \int \Phi^{-1}(t)F(t)\,dt = \begin{pmatrix} -\frac{7}{3}e^{t} \\ -\frac{1}{3}e^{-5t} \end{pmatrix} \quad (2\,\text{pts})$$

$$X_p = \Phi(t)\int \Phi^{-1}(t)F(t)\,dt \quad (2\,\text{pts})$$

$$= \begin{pmatrix} 0 & 3e^{5t} \\ e^{-t} & 2e^{5t} \end{pmatrix} \begin{pmatrix} -\frac{7}{3}e^{t} \\ -\frac{1}{3}e^{-5t} \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ -3 \end{pmatrix} \quad (1\,\text{pt})$$

16. Given $A = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}$.

(a) [**6 points**] Compute e^{At}

$$A^2 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad (1\text{ pt})$$

$$A^3 = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (1\text{ pt})$$

$$e^{At} = I + At + A^2\frac{t^2}{2!} + \cancel{A^3\frac{t^3}{3!}}^{\to 0} + \cdots \quad (2\text{ pts})$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -t & t & t \\ -t & 0 & t \\ -t & t & t \end{pmatrix} + \begin{pmatrix} -\frac{1}{2}t^2 & 0 & \frac{1}{2}t^2 \\ 0 & 0 & 0 \\ -\frac{1}{2}t^2 & 0 & \frac{1}{2}t^2 \end{pmatrix} \quad (1\text{ pt})$$

$$= \begin{pmatrix} 1-t-\frac{t^2}{2} & t & t+\frac{t^2}{2} \\ -t & 1 & t \\ -t-\frac{t^2}{2} & t & 1+t+\frac{t^2}{2} \end{pmatrix} \quad (1\text{ pt})$$

(b) [**2 points**] Use part(a) to give the general solution of the system $X' = AX$.

The general solution is $X = e^{At} \cdot C$ (1 pt)

$$X = \begin{pmatrix} 1-t-\frac{t^2}{2} & t & t+\frac{t^2}{2} \\ -t & 1 & t \\ -t-\frac{t^2}{2} & t & 1+t+\frac{t^2}{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \quad (1\text{ pt})$$