King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam II - Term 163

Duration: 120 minutes

Name: Key ID Number: ______

Section Number: ______ Serial Number: ______

Class Time: ______ Instructor's Name: ______

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 7 Problems)
5. **DE means differential equations.**

Question # Number	**Points**	**Maximum Points**
1		10
2		18
3		10
4		16
5		15
6		16
7		15
Total		100

1. **[10 points]** Given that $y_1(x) = x$ is a solution of

$$(x^2+1)y'' - 2xy' + 2y = 0,$$

find a second solution y_2 such that $\{y_1, y_2\}$ is linearly independent.

We write the DE in standard form

$$y'' - \frac{2x}{x^2+1} y' + \frac{2}{x^2+1} y = 0$$

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx \quad (3\text{ pts})$$

where $P(x) = \dfrac{-2x}{x^2+1}$ (2 pts)

$$\Rightarrow y_2(x) = x \int \frac{e^{\int \frac{2x}{x^2+1}dx}}{x^2}\,dx$$

$$= x \int \frac{e^{\ln(x^2+1)}}{x^2}\,dx \quad (2\text{ pts})$$

$$= x \int \frac{x^2+1}{x^2}\,dx \quad (1\text{ pt})$$

$$= x \int 1 + \frac{1}{x^2}\,dx$$

$$= x\left[x - \frac{1}{x}\right] \quad (2\text{ pts})$$

$$= x^2 - 1$$

2. **[6+6+6 points]** Find the general solution of

(a) $2y''' - 7y'' + 7y' - 2y = 0$

The auxiliary Eqn. is $2m^3 - 7m^2 + 7m - 2 = 0$ (1 pt)

$\Rightarrow (m-1)(2m^2 - 5m + 2) = 0$

$$\begin{array}{c|cccc} 1 & 2 & -7 & 7 & -2 \\ & & 2 & -5 & 2 \\ \hline & 2 & -5 & 2 & 0 \end{array}$$

$\Rightarrow (m-1)(2m-1)(m-2) = 0$

$\Rightarrow m = 1, \frac{1}{2}$, and 2. (3 pts)

The general solution is $y = c_1 e^{x} + c_2 e^{\frac{1}{2}x} + c_3 e^{2x}$

(2 pts)

(b) $(D^5 - 16D^3)^2 y = 0$

The auxiliary Eqn. is

$(m^5 - 16m^3)^2 = 0 \Rightarrow \left(m^3(m^2 - 16)\right)^2 = 0$ (1 pt)

$\Rightarrow m = 0$ (order 6), $m = \pm 4$ (order 2) (2 pts)

The general solution is

$y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + c_5 x^4 + c_6 x^5 + (c_7 + c_8 x) e^{-4x} + (c_9 + c_{10} x) e^{4x}$

(1 pt) (1 pt) (1 pt)

(c) $(D^2 + 4D + 8)^2 y = 0$

The auxiliary Eqn. is $(m^2 + 4m + 8)^2 = 0$ (1 pt)

$\Rightarrow m = \dfrac{-4 \pm \sqrt{16 - 32}}{2} = \dfrac{-4 \pm 4i}{2} = -2 \pm 2i$ (order 2)

(2 pts)

The general solution is

$y = (c_1 + c_2 x) e^{-2x} \cos(2x) + (c_3 + c_4 x) e^{-2x} \sin(2x)$.

(3 pts)

3. **[10 points]** Find an annihilator for the given function

$$e^{2x}\cos^2 3x - xe^{2x}\sin 3x$$

(2pts)

$$e^{2x}\left(\frac{1+\cos 6x}{2}\right) - x e^{2x}\sin 3x$$

$$= \frac{1}{2}e^{2x} + \frac{1}{2}e^{2x}\cos 6x - x e^{2x}\sin(3x)$$

An annihilator of $\frac{1}{2}e^{2x}$ is $D-2$ (1pt)

" " " $\frac{1}{2}e^{2x}\cos(6x)$ " $(D-2)^2+36$ (2pts)

" " " $-xe^{2x}\sin(3x)$ " $((D-2)^2+9)^2$ (2pts)

So an annililator of

$$e^{2x}\cos^2 3x - x e^{2x}\sin 3x$$

is

$$(D-2)\left[(D-2)^2+36\right]\left[(D-2)^2+9\right]^2$$ (3pts)

4. **[16 points]** Use the method of undetermined coefficients to find a particular solution of

$$y'' - 3y' + 2y = 6e^{-x} + e^{x} + e^{2x}.$$

(1pt)

$y'' - 3y' + 2y = 0$. The auxiliary Eqn. is $m^2 - 3m + 2 = 0$

$\Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1, m = 2.$

$\Rightarrow y_c = c_1 e^{x} + c_2 e^{2x}$ (2pts)

An annihilator of the right-hand side of the DE is $(D+1)(D-1)(D-2)$. (3pts)

$\Rightarrow (D+1)(D-1)^2(D-2)^2 y = 0$ (1pt)

(1pt)

$\therefore (m+1)(m-1)^2(m-2)^2 = 0 \Rightarrow m = -1, 1, 1, 2, 2.$
(1pt)

$\Rightarrow y = b_1 e^{-x} + b_2 e^{x} + b_3 x e^{x} + b_4 e^{2x} + b_5 x e^{2x}$ (1pt)

$\therefore y_p$ has the form:

$y_p = A e^{-x} + B x e^{x} + C x e^{2x}$ (3pts)

$y_p' = -A e^{-x} + B e^{x} + B x e^{x} + C e^{2x} + 2C x e^{2x}$

$y_p'' = A e^{-x} + 2B e^{x} + B x e^{x} + 4C e^{2x} + 4C x e^{2x}$

substituting in the DE, we get

$A e^{-x} + 2B e^{x} + B x e^{x} + 4C e^{2x} + 4C x e^{2x} + 3A e^{-x} - 3B e^{x} - 3B x e^{x}$
$- 3C e^{2x} - 6C x e^{2x} + 2A e^{-x} + 2B x e^{x} + 2C x e^{2x} = 6 e^{-x} + e^{x} + e^{2x}$

$\Rightarrow 6A e^{-x} - B e^{x} + C e^{2x} = 6 e^{-x} + e^{x} + e^{2x}$

$\therefore A = 1, \quad B = -1, \quad C = 1. \quad y_p = e^{-x} - x e^{x} + x e^{2x}$
(1pt) (1pt) (1pt)

5. **[15 points]** Use the method of variation of parameters to solve

$$y'' + y = \sec x \tan x$$

To find y_c, we solve $y'' + y = 0$

$m^2 + 1 = 0 \Rightarrow m = \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$ (3 pts)

let $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$ (2 pts)

where $y_1(x) = \cos x$, $y_2(x) = \sin x$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \quad (2\text{ pts})$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \end{vmatrix} = -\tan^2 x \quad (1\text{ pt})$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \end{vmatrix} = \tan x \quad (1\text{ pt})$$

$$u_1(x) = \underbrace{\int \frac{W_1}{W}\, dx}_{(1\text{ pt})} = \int -\tan^2 x \, dx$$

$$= \int 1 - \sec^2 x \, dx = x - \tan x \quad (2\text{ pts})$$

$$u_2(x) = \underbrace{\int \frac{W_2}{W}\, dx}_{(1\text{ pt})} = \int \tan x \, dx = \ln|\sec x| \quad (1\text{ pt})$$

$$\therefore y = y_c + y_p = c_1 \cos x + c_2 \sin x + x \cos x - \sin x + \sin x \ln|\sec x| \quad (1\text{ pt})$$

$$= c_1 \cos x + c_3 \sin x + x \cos x + \sin x \ln|\sec x|$$

6. (a) **[8 points]** Solve

$$x^2y'' + 7xy' + 10y = 0, \qquad x > 0$$

This is a C.E. let $y = x^m \Rightarrow y' = m x^{m-1}$ (1pt)

$\Rightarrow y'' = m(m-1)\, x^{m-2}$

substituting in the DE, we get

$m^2 + 6m + 10 = 0$ (3pts)

$\Rightarrow m = \dfrac{-6 \pm \sqrt{36-40}}{2} = \dfrac{-6 \pm 2i}{2} = -3 \pm i$ (2pts)

The general solution is

$y = c_1 x^{-3} \cos(\ln x) + c_2 x^{-3} \sin(\ln x)$

(2 pts)

(b) **[8 points]** Solve

$$x^2y'' - 7xy' + 16y = 0, \qquad x > 0$$

This is a C.E. let $y = x^m \Rightarrow y' = m x^{m-1} \Rightarrow y'' = m(m-1)x^{m-2}$ (1pt)

substituting in the DE, we get

$m^2 - 8m + 16 = 0$ (3pts)

$\Rightarrow (m-4)^2 = 0 \Rightarrow m = 4, 4$ (2pts)

So the general solution is

$y = (c_1 + c_2 \ln x)\, x^4$

(2 pts)

7. **[15 points]** Find two linearly independent power series solutions of

$$2y'' + xy' + y = 0$$

about $x = 0$ and write the region of validity.
(write the first three nonzero terms for each series solution).

Let $y = \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$ (3pts)

Substituting in the DE, we get

$$\sum_{n=2}^{\infty} 2n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0 \text{ (1pt)}$$

$$\Rightarrow 4a_2 + a_0 + \sum_{n=3}^{\infty} 2n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=1}^{\infty} a_n x^n = 0$$

$$\Rightarrow (4a_2 + a_0) + \sum_{n=1}^{\infty} 2(n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} (n+1) a_n x^n = 0$$

$$\Rightarrow (4a_2 + a_0) + \sum_{n=1}^{\infty} \left[2(n+2)(n+1) a_{n+2} + (n+1) a_n\right] x^n = 0 \text{ (3pts)}$$

(1pt) (2pts)

$$\Rightarrow a_2 = -\frac{1}{4} a_0 \quad \text{and} \quad a_{n+2} = \frac{-1}{2(n+2)} a_n, \quad n = 1, 2, 3, \ldots$$

$n = 1 \Rightarrow a_3 = \frac{-1}{6} a_1$, $n = 2 \Rightarrow a_4 = \frac{-1}{8} a_2 = \frac{1}{32} a_0$

$n = 3 \Rightarrow a_5 = \frac{-1}{10} a_3 = \frac{1}{60} a_1$

$n = 4 \Rightarrow a_6 = \frac{-1}{12} a_4 = -\frac{1}{12} \cdot \frac{1}{32} a_0 = \frac{-1}{384} a_0$

$$y(x) = a_0\left(1 - \frac{1}{4}x^2 + \frac{1}{32}x^4 - \frac{1}{384}x^6 \ldots\right) \text{ (2pts)}$$

$$+ a_1\left(x - \frac{1}{6}x^3 + \frac{1}{60}x^5 + \ldots\right.$$

(2pts)

$|x| < \infty$

(1pt)