King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam I - Term 163

Duration: 120 minutes

Name: Key

ID Number: ______________________

Section Number: ______________________

Serial Number: ______________________

Class Time: ______________________

Instructor's Name: ______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 10 pages of problems (Total of 10 Problems)
5. **DE means differential equations**.

Question # Number	Points	Maximum Points
1		10
2		12
3		12
4		12
5		12
6		6
7		12
8		12
9		6
10		6
Total		100

1. (a) [**4 points**] Verify that $x^2y^4 + x^3 - 27 = 0$ is an implicit solution of the DE $4xy^3\frac{dy}{dx} + 2y^4 + 3x = 0, \qquad 0 < x < 3.$

$$x^2y^4 + x^3 - 27 = 0$$

(1 pt) (1 pt) (1 pt)

$$\Rightarrow 2xy^4 + x^2\left(4y^3\frac{dy}{dx}\right) + 3x^2 = 0$$

$$\Rightarrow 2y^4 + 4xy^3\frac{dy}{dx} + 3x = 0 \quad (1\text{pt})$$

or $4xy^3\frac{dy}{dx} + 2y^4 + 3x = 0$.

$\therefore$ $x^2y^4 + x^3 - 27 = 0$ is an implicit solution of the DE.

(b) [**6 points**] Find all singular constant solutions of the DE $\frac{dy}{dx} = y^2 - 4$ given that $y = 2\frac{1+ce^{4x}}{1-ce^{4x}}$ is a one-parameter family of solutions of the differential equations.

(1 pt) Let $y = K \Rightarrow \frac{dy}{dx} = 0$.

substitute in the DE, we get

$K^2 - 4 = 0 \Rightarrow K = \pm 2$.

So the DE has two constant solutions

(2 pts) $y = 2$ and $y = -2$

(1pt) $y = +2$ is not singular: $2\frac{1+ce^{4x}}{1-ce^{4x}} = 2 \Rightarrow 1 + ce^{4x} = 1 - ce^{4x}$

$\Rightarrow 2ce^{4x} = 0$

$\Rightarrow \boxed{c = 0}$

(2 pts) $y = -2$ is singular:

$2\frac{1+ce^{4x}}{1-ce^{4x}} = -2 \Rightarrow 1 + \cancel{ce^{4x}} = -1 + \cancel{ce^{4x}}$ impossible.

2. **[12 points]** Solve: $\dfrac{dy}{dx} = \dfrac{xy + 3x - y - 3}{xy - 2x + 4y - 8}$, $y \neq 2, y \neq -3$.

$$\frac{dy}{dx} = \frac{x(y+3) - (y+3)}{x(y-2) + 4(y-2)}$$

(2 pts)

$$\Rightarrow \frac{dy}{dx} = \frac{(y+3)(x-1)}{(y-2)(x+4)}$$

Separable DE.

(2 pts) (2 pts)

$$\Rightarrow \frac{y-2}{y+3}\,dy = \frac{x-1}{x+4}\,dx \Rightarrow \int \frac{y-2}{y+3}\,dy = \int \frac{x-1}{x+4}\,dx$$

(1 pt) (1 pt)

$$\Rightarrow \int \left(1 - \frac{5}{y+3}\right) dy = \int \left(1 - \frac{5}{x+4}\right) dx$$

(1 pt) (1 pt) (1 pt) (1 pt)

$$\Rightarrow y - 5\ln|y+3| = x - 5\ln|x+4| + C$$

or

$$\Rightarrow (y - x) + 5\ln\left|\frac{x+4}{y+3}\right| = C$$

3. **[12 points]** Show that the given differential equation is exact, and then solve it:

$$(1 + 2x - y^3)\,dx + (2y - 3xy^2)\,dy = 0.$$

$M(x,y) = 1+2x-y^3$

$N(x,y) = 2y-3xy^2$

(3pts) $\frac{\partial M}{\partial y} = -3y^2 = \frac{\partial N}{\partial x}$ $\therefore$ the DE is Exact.

So, there is a function $f(x,y)$ such that

(2pts) $\frac{\partial f}{\partial x} = 1+2x-y^3$ and $\frac{\partial f}{\partial y} = 2y-3xy^2$ (2pts)

$\frac{\partial f}{\partial x} = 1+2x-y^3 \Rightarrow f(x,y) = x + x^2 - y^3x + g(y)$

$\frac{\partial f}{\partial y} = \cancel{-3xy^2} + g'(y) = 2y - \cancel{3xy^2}$ (2pts)

$\Rightarrow g(y) = y^2 + c_1$ (1pt)

$\therefore f(x,y) = x + x^2 - xy^3 + y^2 + c_1$

The solution of the DE is

$x + x^2 - xy^3 + y^2 + c_1 = c_2$

or

$x + x^2 - xy^3 + y^2 = C$. (2pts)

4. **[12 points]** Solve the linear DE

$$\sin y\, dx + 2(x - 3\sin y)\cos y\, dy = 0, \quad x\left(\frac{\pi}{2}\right) = \frac{1}{2}.$$

(2 pts) $\sin y \dfrac{dx}{dy} + (2\cos y)x = 6 \sin y \cos y$

(1 pts) (*) $\dfrac{dx}{dy} + 2\dfrac{\cos y}{\sin y} x = 6\cos y$ (Linear in x)

$$u(y) = e^{2\int \frac{\cos y}{\sin y} dy} = e^{2\ln|\sin y|} = e^{\ln(\sin^2 y)}, \; 0 < y < \pi$$

(3 pts) $\Rightarrow u(y) = \sin^2 y$

multiply (*) by $\sin^2 y$ to get

(2pts) $\dfrac{d}{dy}\left(\sin^2 y\, x\right) = 6\sin^2 y \cos y$

(1 pts) $\Rightarrow \sin^2 y \cdot x = 2\sin^3 y + C$

$\Rightarrow x = 2\sin y + C\csc^2 y$

$x\left(\frac{\pi}{2}\right) = \frac{1}{2} \Rightarrow \frac{1}{2} = C + 2$

$\therefore C = -\frac{3}{2}$ (2 pts)

The solution is

$x = 2\sin y - \frac{3}{2}\csc^2 y$. (1 pt)

5. (a)[**6 points**] Use a suitable substitution to transform the given **DE** in to a linear **DE** and write down the new equation you obtained
(Do not solve the new equation)

$$y' = \frac{y(1+x-6y^2)}{2x}$$

(1 pt) $2x\frac{dy}{dx} = (1+x)y - 6y^3$

(1 pt) $\Rightarrow \frac{dy}{dx} - \frac{(1+x)}{2x}y = -3x^{-1}y^3$

Bernoulli DE with $n=3$

$\Rightarrow y^{-3}\frac{dy}{dx} - \frac{(1+x)}{2x}y^{-2} = -3x^{-1}$

Let $u = y^{-2}$ (1 pt)

$\frac{du}{dx} = -2y^{-3}\frac{dy}{dx}$ (1 pt)

substitute in the DE to get

(2 pts) $-\frac{1}{2}\frac{du}{dx} - \frac{(1+x)}{2x}u = -3x^{-1}$

or $\frac{du}{dx} + (1+\frac{1}{x})u = \frac{6}{x}$ which is Linear in u.

(b) [**6 points**]Use a suitable substitution to transform the given **DE** in to a separable **DE** and write down the new equation you obtained
(Do no solve the new equation)

$$\frac{dy}{dx} = (x+y)e^{x+y}$$

(2 pts) Let $u = x+y \Rightarrow \frac{du}{dx} = 1 + \frac{dy}{dx}$ (1 pt)

substitute in the DE to get

$\frac{du}{dx} - 1 = ue^u$ (1 pt)

$\Rightarrow \frac{du}{dx} = 1 + ue^u$ (2 pts)

$\Rightarrow \frac{du}{1+ue^u} = dx$ which is separable.

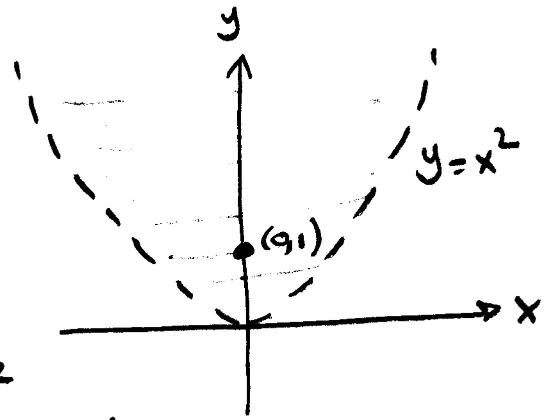
6. [**6 points**] Does the IVP

$$\frac{dy}{dx} = \sqrt{y - x^2}, \; y(0) = 1$$

have a unique solution? **Justify your answer.**

(2 pts) $f(x,y) = \sqrt{y-x^2}$ is continuous if $y \geq x^2$

(2 pts) $\dfrac{\partial f}{\partial y} = \dfrac{1}{2\sqrt{y-x^2}}$ is continuous if $y > x^2$



Since $f(x,y)$ and $\frac{\partial f}{\partial y}$ are both continuous in a rectangle containing the point $(0,1)$ in its interior, then by Existence and Uniqueness theorem the IVP has a unique solution

(2 pts)

7. **[12 points]** At 1:00 pm, an object whose temperature is $100^\circ C$ is taken outside where the air temperature is $20^\circ C$. At 1:20 pm, the temperature of the object is $60^\circ C$. At what time does the temperature of the object become $30^\circ C$?

Let $T(t)$ be the temperature of the object at any time t.

(2 pts)

$$\frac{dT}{dt} = K(T - T_m), \quad T(0) = 100^\circ C$$

$T_m = 20^\circ C \leftarrow$ (1 pt)

$T(20) = 60^\circ C$

$T(t) = 30, \ t = ?$

Solving the DE, we get

$$T(t) - T_m = c\, e^{Kt} \quad \text{(2 pts)}$$

$$T(t) - 20 = c\, e^{Kt}$$

$$T(0) = 100 \Rightarrow 80 = c \quad \text{(2 pts)}$$

$$\Rightarrow T(t) - 20 = 80\, e^{Kt}$$

$$T(20) = 60 \Rightarrow 40 = 80\, e^{20K}$$

$$\Rightarrow e^{20K} = \frac{1}{2} \Rightarrow 20K = \ln\left(\frac{1}{2}\right)$$

$$\text{or } K = \frac{1}{20}\ln\left(\frac{1}{2}\right) \quad \text{(2 pts)}$$

$$\Rightarrow T(t) = 20 + 80\, e^{\frac{t}{20}\ln\left(\frac{1}{2}\right)}$$

$$T(t) = 30 \Rightarrow 10 = 80\, e^{\ln\left(\frac{1}{2}\right)^{t/20}}$$

$$\text{or } \left(\frac{1}{2}\right)^{t/20} = \left(\frac{1}{2}\right)^3 \Rightarrow \frac{t}{20} = 3 \Rightarrow t = 60 \quad \text{(2 pts)}$$

So, at 2:00 pm $T(t) = 30$. (1 pt)

8. **[12 points]** Show that $\{\sin(x^2), \cos(x^2)\}$ form a fundamental set of solutions of the DE

$$xy'' - y' + 4x^3 y = 0, \qquad x > 0.$$

step I : we need to show that $y_1(x) = \sin(x^2)$ and $y_2(x) = \cos(x^2)$ are both solutions of the DE.

(3pts) $y_1(x) = \sin(x^2) \Rightarrow y_1'(x) = 2x\cos(x^2)$

$y_1''(x) = 2\cos(x^2) - 4x^2\sin(x^2)$

substituting we get

$$L.H.S = \cancel{2x\cos(x^2)} - \cancel{4x^3\sin(x^2)} - \cancel{2x\cos(x^2)} + \cancel{4x^3\sin(x^2)}$$

$$= 0 = R.H.S$$

(3 pts) $y_2(x) = \cos(x^2) \Rightarrow y_2'(x) = -2x\sin(x^2)$

$\Rightarrow y_2''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$

Substituting we get

$$L.H.S = \cancel{-2x\sin(x^2)} - \cancel{4x^3\cos(x^2)} + \cancel{2x\sin(x^2)} + \cancel{4x^3\cos(x^2)}$$

$$= 0 = R.H.S$$

step II : show that $y_1(x)$ and $y_2(x)$ are L.I.

$$W(y_1, y_2) = \begin{vmatrix} \sin(x^2) & \cos(x^2) \\ 2x\cos(x^2) & -2x\sin(x^2) \end{vmatrix} = -2x\sin^2(x^2) - 2x\cos^2(x^2)$$

(2 pts) (2 pts)

$= -2x \neq 0$ for $x > 0$ (1 pt)

So, $y_1(x)$ and $y_2(x)$ are L.I. (1pt)

9. **[6 points]** Given that $y = c_1 \cos 2x + c_2 \sin 2x$ is a two-parameter family of solutions of the DE $y'' + 4y = 0$. Determine whether a member of the family can be found so that $y(0) = 1$ and $y'(\pi) = 4$.

$$y(0) = 1 \Rightarrow c_1 \underbrace{\cos(0)}_{1} + c_2 \underbrace{\sin(0)}_{0} = 1$$

$$\Rightarrow c_1 = 1 \quad (2\text{pts})$$

$$y'(x) = -2c_1 \sin(2x) + 2c_2 \cos(2x) \quad (1\text{pt})$$

$$y'(\pi) = 4 \Rightarrow 2c_2 = 4$$

$$\Rightarrow c_2 = 2 \quad (2\text{pts})$$

$\therefore$ The solution is $y = \cos(2x) + 2\sin(2x)$

(1pt)

10. **[6 points]** Without the use of the Wronskian, show that $f_1(x) = 2e^x$, $f_2(x) = 3 - 5e^x$ and $f_3(x) = 4$ are linearly dependent on the interval $(-\infty, \infty)$.

Note that

$$f_2(x) = 3 - 5e^x$$

(2pts) (2pts)

$$= \frac{3}{4}(4) + \left(\frac{-5}{2}\right)(2e^x)$$

$$= \frac{3}{4} f_3(x) - \frac{5}{2} f_1(x) \qquad (1pt)$$

So, the three functions are linearly dependent (1 pt)