

King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 201- Final Exam – 2016–2017 (163)

Allowed Time: 180 minutes

Name: _____ ID #: _____

Section #: _____ Serial Number: _____

Instructions:

1. Make sure that you have 7 questions of written problems and 14 multiple choice questions (MCQ)
2. Show all your work for written problems. No points for answers without justification.
3. Calculators, Mobiles and Smart Devices are not allowed.

Written Problems:

Question #	Grade	Maximum Points
1		12
2		10
3		10
4		10
5		08
6		10
7		10
Total:		70

MCQ Problems:

Circle your answer in the table below: 5 points for each correct answer

Question #	Answer	Grade
1	(a)	
2	(a)	
3	(a)	
4	(a)	
5	(a)	
6	(a)	
7	(a)	
8	(a)	
9	(a)	
10	(a)	
11	(a)	
12	(a)	
13	(a)	
14	(a)	
Total:		

1. [12 points] Find the absolute maximum and minimum values of

$$f(x, y) = 2x^2 + y^4 + 4xy$$

on the closed triangular region R bounded by the lines $y = -2x$, $y = 0$, $x = 2$.

Critical points: $f_x = 4x + 4y = 0 \Rightarrow y = -x$
 $f_y = 4y^3 + 4x = 0 \Rightarrow y^3 + x = 0$
 $\Rightarrow -x^3 + x = 0$
 $\Rightarrow x = 0, \pm 1$

CP $(x, y) = (0, 0), (1, -1), (-1, 1)$

CP inside the region $(1, -1)$

On the line: $y = 0, 0 \leq x \leq 2$

$$f(x, 0) = 2x^2 \Rightarrow f'(x, 0) = 4x = 0 \Rightarrow x = 0$$

Pts $(0, 0), (2, 0)$

On the line segment: $x = 2, -4 \leq y \leq 0$

$$f(2, y) = 8 + y^4 + 8y$$

$$f'(2, y) = 4y^3 + 8 = 0 \Rightarrow y = (-2)^{1/3}$$

Pts $(2, (-2)^{1/3}), (2, 0), (2, -4)$

on the line: $y = -2x, 0 \leq x \leq 2$

$$f(x, -2x) = 2x^2 + 16x^4 - 8x^2 = 16x^4 - 6x^2$$

$$f'(x, -2x) = 4x(16x^2 - 3) = 0 \Rightarrow x = 0, \pm \frac{\sqrt{3}}{4}$$

$$\Rightarrow x = 0, \frac{\sqrt{3}}{4} \in [0, 2]$$

Pts $(0, 0), (2, -4), (\frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{2})$

f :

$$f(0, 0) = 0$$

$$f(2, 0) = 8$$

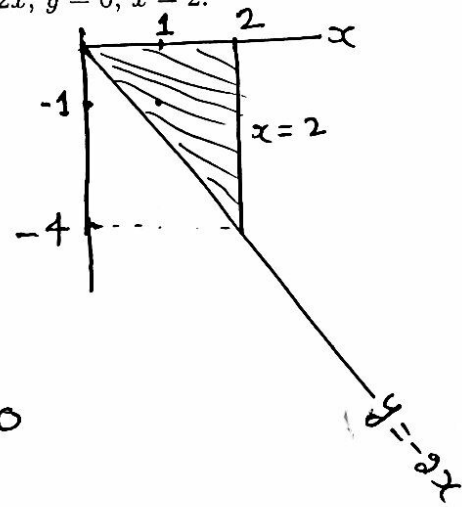
$$f(2, -4) = 232$$

$$f(1, -1) = -1$$

$$f(2, (-2)^{1/3}) = 8 - 6(2)^{1/3}$$

$$f(\frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{2}) = -\frac{9}{16}$$

— absolute maximum
 — absolute minimum



2. [10 points] Use Lagrange multipliers to find maximum and minimum values of the function

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

like solve the system

$$\begin{aligned} \nabla f &= \lambda \nabla g \\ g &= R \end{aligned} \Rightarrow \langle 1, -2, 5 \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$x^2 + y^2 + z^2 = 30 \quad (*)$$

$$\Rightarrow x = \frac{1}{2\lambda}$$

$$y = -\frac{1}{\lambda}$$

$$z = \frac{5}{2\lambda}$$

$$(*) \text{ gives } \left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = 30$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = 30$$

$$\Rightarrow 4\lambda^2 = 1$$

$$\Rightarrow \lambda = \pm \frac{1}{2}$$

$$\text{For } \lambda = \frac{1}{2}, \quad x = 1, \quad y = -2, \quad z = 5$$

$$\text{" } \lambda = -\frac{1}{2}, \quad x = -1, \quad y = 2, \quad z = -5$$

$$f(1, -2, 5) = 30 \quad \text{max.}$$

$$f(-1, 2, -5) = -30 \quad \text{min}$$

3. [10 points] Evaluate $\int_0^1 \int_x^1 e^{x/y} dy dx$.

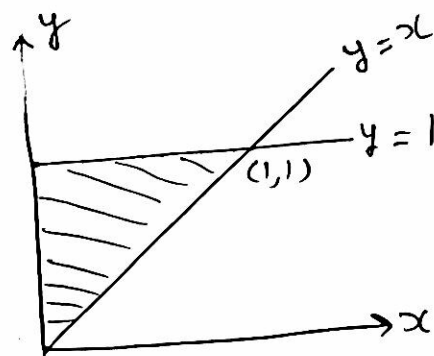
$$= \int_0^1 \int_{x=0}^y e^{\frac{x}{y}} dx dy$$

$$= \int_0^1 \left(y e^{x/y} \right)_0^y dy$$

$$= \int_0^1 (e-1) y dy$$

$$= (e-1) \frac{y^2}{2} \Big|_0^1$$

$$= \frac{e-1}{2}$$

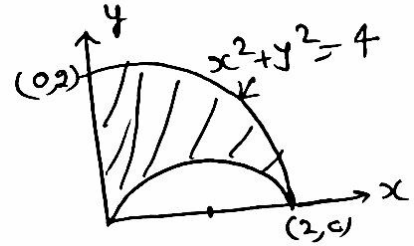


Let $\frac{x}{y} = u$
 $dx = y du$

4. [10 points] Evaluate $\iint_R y \, dA$, where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

$$x^2 + y^2 = 4 \Rightarrow r = 2$$

$$x^2 + y^2 = 2x \Rightarrow r = 2 \cos \theta$$



$$\iint_R y \, dA$$

$$= \int_0^{\pi/2} \int_{2 \cos \theta}^2 r \sin \theta \cdot r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3} \right)_{2 \cos \theta}^2 \cdot \sin \theta \, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3} (8 - 8 \cos^3 \theta) \sin \theta \, d\theta$$

$$= \frac{1}{3} \left[-8 \cos \theta + \frac{8}{4} \cos^4 \theta \right]_0^{\pi/2}$$

$$= \frac{1}{3} (8 - 2) = \frac{6}{3} = 2.$$

5. [8 points] Evaluate $\int_1^4 \int_y^4 \int_0^z \frac{z}{x^2 + z^2} dx dz dy$.

$$= \int_1^4 \int_y^4 z \cdot \frac{1}{z} \tan^{-1} \frac{x}{z} \Big|_0^z dz dy$$

$$= \int_1^4 \int_y^4 (\tan^{-1} 1 - \tan^{-1} 0) dz dy$$

$$= \frac{\pi}{4} \int_1^4 \int_y^4 dz dy$$

$$= \frac{\pi}{4} \int_1^4 (4 - y) dy$$

$$= \frac{\pi}{4} \left[4y - \frac{y^2}{2} \right]_1^4$$

$$= \frac{\pi}{4} \left[16 - 8 - 4 + \frac{1}{2} \right]$$

$$= \frac{\pi}{4} \left[4 + \frac{1}{2} \right] = \frac{9\pi}{8}$$

6. [10 points] Find the volume of the solid bounded by the paraboloids $z = 5 - x^2 - y^2$ and $z = 4x^2 + 4y^2$.

The paraboloids intersect when

$$5 - x^2 - y^2 = 4x^2 + 4y^2$$

$$\Rightarrow x^2 + y^2 = 1$$

$$\text{Volume} = 4 \int_0^{\pi/2} \int_0^1 \int_{4r^2}^{5-r^2} dz \cdot r dr d\theta$$

$$= 4 \int_0^{\pi/2} \int_0^1 (5r - r^3 - 4r^3) dr d\theta$$

$$= 4 \int_0^{\pi/2} \left(5 \frac{r^2}{2} - \frac{5}{4} r^4 \right) \Big|_0^1 d\theta$$

$$= 4 \int_0^{\pi/2} \frac{5}{4} d\theta$$

$$= \frac{5\pi}{2}$$

or

$$\int_0^{2\pi} \int_0^1 \int_{4r^2}^{5-r^2} dz \cdot r dr d\theta$$

$$= \int_0^{2\pi} \frac{5}{4} d\theta$$

$$= \frac{5\pi}{2}$$

7. [10 points] Use spherical coordinates to find the volume of the part of the ball $x^2 + y^2 + z^2 \leq 4$ that lies between the cones

$$2z = \sqrt{2x^2 + 2y^2} \text{ and } 2z = \sqrt{x^2 + y^2}.$$

In spherical the cones are given by

$$(i) \quad z = \frac{1}{2} \sqrt{2x^2 + 2y^2} = \frac{\sqrt{2}}{2} \sqrt{x^2 + y^2} = \frac{\sqrt{2}}{2} r$$

$$\frac{z}{r} = \frac{\sqrt{2}}{2}$$

$$\cos \phi = \frac{\sqrt{2}}{2} \Rightarrow \phi = \frac{\pi}{4}$$

$$(ii) \quad z = \frac{1}{2} \sqrt{x^2 + y^2} \Rightarrow \frac{z}{r} = \frac{1}{2}$$

$$\cos \phi = \frac{1}{2}$$

$$\phi = \frac{\pi}{3}$$

$$V = \int_0^{2\pi} \int_{\frac{\pi}{4}}^{\pi/3} \int_0^2 \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \left[\frac{1}{3} \rho^3 \right]_0^2 \left[-\cos \phi \right]_{\frac{\pi}{4}}^{\pi/3}$$

$$= 2\pi \cdot \frac{8}{3} \left(-\frac{1}{2} + \frac{\sqrt{2}}{2} \right)$$

$$= \frac{8\pi}{3} (\sqrt{2} - 1)$$

1. The Cartesian equation of the curve defined by

$$x = \cos(\pi - t), y = \sin(\pi - t), 0 \leq t \leq \pi$$

is

t	x	y
0	-1	0
$\frac{\pi}{2}$	0	1
π	1	0

- (a) $x^2 + y^2 = 1, -1 \leq x \leq 1, 0 \leq y \leq 1$
 (b) $x^2 + y^2 = 1, 0 \leq x \leq 1, 0 \leq y \leq 1$
 (c) $x^2 + y^2 = 1, -1 \leq x \leq 1, -1 \leq y \leq 1$
 (d) $x^2 - y^2 = 1, -1 \leq x \leq 1, 0 \leq y \leq 1$
 (e) $x^2 + y^2 = \pi^2, 0 \leq x \leq 1, 0 \leq y \leq 1$

2. The graph of $r = 4 \tan \theta \sec \theta$ is

- (a) a parabola
 (b) a circle
 (c) an ellipse
 (d) a straight line
 (e) a hyperbola

$$\begin{aligned} r &= 4 \frac{\sin \theta}{\cos \theta} \cdot \frac{1}{\cos \theta} \\ &= 4 \frac{y}{r} \cdot \frac{r}{x} \cdot \frac{r}{x} \\ y &= \frac{x^2}{4} \end{aligned}$$

3. Length of the polar curve

$$r = \sqrt{1 + \sin 2\theta}, \quad 0 \leq \theta \leq \sqrt{2}\pi \quad \frac{dr}{d\theta} = \frac{\cos 2\theta}{\sqrt{1 + \sin 2\theta}}$$

is equal to:

(a) 2π

(b) 1

(c) π

(d) 4π

(e) 4

$$\begin{aligned} s &= \int_0^{\sqrt{2}\pi} \sqrt{1 + \sin 2\theta + \frac{\cos^2 2\theta}{1 + \sin 2\theta}} d\theta \\ &= \int_0^{\sqrt{2}\pi} \sqrt{\frac{1 + \sin^2 2\theta + 2\sin 2\theta + \cos^2 2\theta}{1 + \sin 2\theta}} d\theta \\ &= \int_0^{\sqrt{2}\pi} \sqrt{\frac{1 + 1 + 2\sin 2\theta}{1 + \sin 2\theta}} d\theta \\ &= \int_0^{\sqrt{2}\pi} \sqrt{2} d\theta = \sqrt{2} \cdot \sqrt{2}\pi \\ &= 2\pi \end{aligned}$$

4. The quadratic surface $x^2 - y^2 + z^2 - 2x + 2y - 4z + 2 = 0$ represents

$$(x-1)^2 - (y-1)^2 + (z-2)^2 = 2$$

(a) a hyperboloid of one sheet

(b) a hyperboloid of two sheets

(c) an ellipsoid

(d) a cone

(e) a paraboloid

5. The value of k for which the vectors $\vec{a} = \langle 4, -3, 1 \rangle$,
 $\vec{b} = \langle 2, 1, 3 \rangle$ and $\vec{c} = \langle k, 0, 2 \rangle$ lie in the same plane is

(a) $k = 2$

(b) $k = 1$

(c) $k = -1$

(d) $k = -2$

(e) $k = 0$

$$\begin{vmatrix} 4 & -3 & 1 \\ 2 & 1 & 3 \\ k & 0 & 2 \end{vmatrix} = 0$$

$$8 + 3(4 - 3k) - k = 0$$

$$20 - 9k - k = 0$$

$$20 - 10k = 0$$

$$k = 2$$

6. If the angle between two unit vectors \vec{u}_1 and \vec{u}_2 is $\frac{\pi}{4}$,
 then $|2\vec{u}_1 - \vec{u}_2|^2 =$

(a) $5 - 2\sqrt{2}$

(b) $3 - 2\sqrt{2}$

(c) $1 + 2\sqrt{2}$

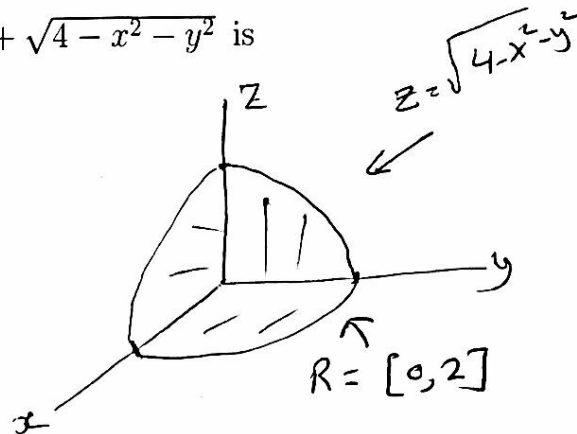
(d) $5 + 2\sqrt{2}$

(e) $3 + 2\sqrt{2}$

$$\begin{aligned} |2\vec{u}_1 - \vec{u}_2|^2 &= (2\vec{u}_1 - \vec{u}_2) \cdot (2\vec{u}_1 - \vec{u}_2) \\ &= 4|\vec{u}_1|^2 - 4\vec{u}_1 \cdot \vec{u}_2 + |\vec{u}_2|^2 \\ &= 4 - 4|\vec{u}_1||\vec{u}_2|\cos\frac{\pi}{4} + 1 \\ &= 5 - 4\frac{\sqrt{2}}{2} \\ &= 5 - 2\sqrt{2} \end{aligned}$$

7. The range of the function $F(x, y) = 1 + \sqrt{4 - x^2 - y^2}$ is

- (a) $[1, 3]$
- (b) $[0, 2]$
- (c) $[-1, 0]$
- (d) $[2, 5]$
- (e) $[1, \infty)$



8. If $L = \lim_{(x,y) \rightarrow (0,0)} \frac{e^{1+x^2+y^2} - e}{x^2 + y^2}$, then

- (a) $L = e$
- (b) L does not exist
- (c) $L = 0$
- (d) $L = 1$
- (e) $L = 2e$

$$\begin{aligned}
 & \text{Put } r^2 = x^2 + y^2 \\
 L &= \lim_{r \rightarrow 0^+} \frac{e^{1+r^2} - e}{r^2} \\
 &= \lim_{r \rightarrow 0^+} \frac{2r e^{1+r^2}}{2r} \\
 &= \lim_{r \rightarrow 0^+} e^{1+r^2} \\
 &= e
 \end{aligned}$$

9. Let $F(x, y, z) = x^2 \sqrt{y+z} + \frac{2y}{xz}$.

The value of $F_{xy} + F_{xz}$ at the point $(-1, 0, 1)$ is

(a) -4

$$F_x = 2x\sqrt{y+z} - \frac{2y}{x^2z}$$

(b) -3

$$F_{xy} = \frac{2x}{2\sqrt{y+z}} - \frac{2}{x^2z}$$

(c) -1

$$F_{xz} = \frac{2x}{2\sqrt{y+z}} + \frac{2y}{x^2z^2}$$

(d) 2

(e) 0

$$F_{xy} + F_{xz} \Big|_{(-1, 0, 1)} = -1 - 2 - 1 + 0 = -4$$

10. Let $z = xy^2 + x^2y$, $x = r + s \cos t$, $y = s + r \sin t$.

The value of $\frac{\partial z}{\partial t}$, when $r = 2$, $s = 1$, $t = \pi$, is equal to

(a) -6

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

(b) -3

$$= (y^2 + 2xy)(-s \sin t) + (2xy + x^2)(r \cos t)$$

(c) 1

(d) 2

(e) 5

$$\frac{\partial z}{\partial t} \Big|_{(2, 1, \pi)} = 0 + (2+1)(2 \cos \pi) = (3)(-2) = -6$$

$$x = 2 + 1 \cos \pi = 2 - 1 = 1$$

$$y = 1 + 2 \sin \pi = 1 + 0 = 1$$

11. The directional derivative of $f(x, y, z) = xe^y + ye^z + ze^x$ at the point $(1, 1, 1)$ in the direction of the vector $\vec{v} = \langle 5, 1, -2 \rangle$ is equal to

(a) $\frac{8e}{\sqrt{30}}$

(b) $\frac{2e}{\sqrt{30}}$

(c) $\frac{e}{\sqrt{30}}$

(d) $\frac{-3e}{\sqrt{30}}$

(e) $\frac{-4e}{\sqrt{30}}$

$$\nabla f = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x \rangle$$

$$\nabla f(1, 1, 1) = \langle e + e, e + e, e + e \rangle$$

$$= 2e \langle 1, 1, 1 \rangle$$

$$\nabla f \cdot \hat{u} = 2e \langle 1, 1, 1 \rangle \cdot \frac{\langle 5, 1, -2 \rangle}{\sqrt{25+1+4}}$$

$$= \frac{2e(5+1-2)}{\sqrt{30}} = \frac{8e}{\sqrt{30}}$$

12. The function $f(x, y) = 1 + 2x + 4y - x^2 - 2y^2$ has one critical point (a, b) and

- (a) f has a local maximum at (a, b)
- (b) f has a local minimum at (a, b)
- (c) f has a saddle point at (a, b)
- (d) $f(a, b) = 8$
- (e) $f_{xy}(a, b) = 3$

$$f_x = 2 - 2x = 0 \Rightarrow x = 1$$

$$f_y = 4 - 4y = 0 \Rightarrow y = 1$$

$$f_{xx} = -2$$

$$f_{xy} = 0$$

$$f_{yy} = -4$$

$$D = 8 > 0, f_{xx} < 0$$

13. The value of the double integral

$$\int_0^2 \int_0^{y^2} 3y^3 e^{xy} dx dy$$

is equal to:

- (a) $e^8 - 9$
 (b) $e - 1$
 (c) $e^6 - 5$
 (d) $e^4 - 5$
 (e) $e^8 - 8$

$$\begin{aligned} & \text{Handwritten solution for problem 13:} \\ & xy = u \\ & dx = \frac{du}{y} \\ & \int_0^2 \int_0^{y^2} 3y^3 \frac{e^{xy}}{y} dy dx \\ & = \int_0^2 \left[3y^2 \frac{e^{xy}}{y} \right]_0^{y^2} dy \\ & = \int_0^2 (3y^2 e^{y^3} - 3y^2) dy \\ & = \left[e^{y^3} - y^3 \right]_0^2 \\ & = e^8 - 8 - (e^0 - 0) \\ & = e^8 - 9 \end{aligned}$$

14. If D is the triangular region with vertices $(0, 1)$, $(1, 2)$, $(4, 1)$, then $\iint_D y dA =$

(a) $\frac{8}{3}$
 $L_1: y - 1 = \frac{2-1}{1-0}(x-0)$
 $y - 1 = x$

(b) $\frac{5}{3}$
 $L_2: y - 2 = \frac{1-2}{4-1}(x-1)$
 $y - 2 = -\frac{1}{3}(x-1)$

(c) $\frac{1}{3}$
 $3y - 6 = 1 - x \Rightarrow x = 7 - 3y$

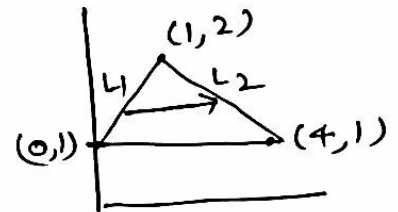
(d) $\frac{10}{3}$
 $\iint_D y dA = \int_{y=1}^2 \int_{y-1}^{7-3y} y dx dy$

(e) $\frac{2}{3}$
 $= \int_1^2 y [7 - 3y - y + 1] dy = \int_1^2 y (8 - 4y) dy$

$$= \left[8 \frac{y^2}{2} - 4 \frac{y^3}{3} \right]_1^2$$

$$= \frac{32}{2} - \frac{32}{3} - \frac{8}{2} + \frac{4}{3}$$

$$= \frac{24}{2} - \frac{28}{3} = 12 - \frac{28}{3} = \frac{36 - 28}{3} = \frac{8}{3}$$



Q	MM	V1	V2	V3	V4
1	a	a	a	b	c
2	a	a	e	e	c
3	a	d	b	a	c
4	a	d	d	c	b
5	a	d	a	b	c
6	a	b	e	b	d
7	a	a	e	e	c
8	a	c	d	e	e
9	a	d	e	b	a
10	a	a	c	d	c
11	a	d	a	c	b
12	a	c	c	c	d
13	a	e	c	d	b
14	a	d	b	e	c