Math201.01, Quiz #3, Term 162

Name: Solutions

ID #: **Serial #:**

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1. [5 points] Find the local maximum and minimum values and saddle points of

$$f(x,y) = x^3y + 12x^2 - 8y.$$

• $\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2y + 24x = 0 & \sim (1) \\ x^3 - 8 = 0 & \sim (2) \end{cases}$ (1)

(1)

$(2) \Rightarrow (x-2)(\underbrace{x^2+2x+4}_{\text{complex}}) = 0$

$\Rightarrow x = 2$, complex

$\overset{(1)}{\Rightarrow} 12y + (24)(2) = 0 \Rightarrow 12(y+4) = 0 \Rightarrow y = -4 \Rightarrow \boxed{(x,y) = (2,-4)}$

• $f_{xx}(x,y) = 6xy + 24$; $f_{yy}(x,y) = 0$; $f_{xy}(x,y) = 3x^2$ (2)

$D(x,y) = f_{xx}\cdot f_{yy} - (f_{xy})^2 = 0 - (3x^2)^2 = -9x^4$

$D(2,-4) = -9(2)^4 = -144 < 0 \Rightarrow f$ has a Saddle point at $(2,-4)$

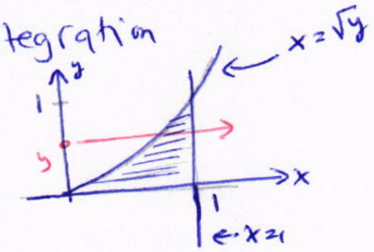
(0.5) (0.5)

• $f(2,-4) = 8(-4) + 12(4) - 8(-4)$
$= 4(-8+12+8)$
$= 48$

2. [4 points] Evaluate $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\; dx\, dy$.

. Not easy to Integrate; Reverse the order of Integration

$R = \{(x,y) : \sqrt{y} \le x \le 1,\ 0 \le y \le 1\}$ Type II



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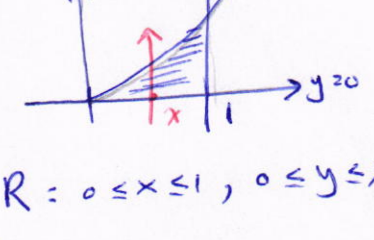

$R: 0 \le x \le 1,\ 0 \le y \le x^2$

• $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\; dx\, dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\; dy\, dx$ (2)

$= \int_0^1 \sqrt{x^3+1}\cdot y \Big]_{y=0}^{y=x^2} dx$ (0.5)

$= \int_0^1 \sqrt{x^3+1}\cdot x^2\; dx$ (0.5)

$= \frac{1}{3}\cdot\frac{2}{3}(x^3+1)^{3/2}\Big]_0^1$ (0.5)

$= \frac{2}{9}(2^{3/2}-1) = \frac{2}{9}(\sqrt{8}-1) = \frac{2}{9}(2\sqrt{2}-1)$

(0.5)

3. [6 points] Use Lagrange Multipliers to find the extreme values of $f(x,y) = x^2y$ subject to the constraint $x^2 + y^2 = 1$.

- $f(x,y) = x^2 y$, $g(x,y) = x^2+y^2-1$
- Solve the System

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Longrightarrow \begin{cases} 2xy = \lambda(2x) & \text{—— (1)} \\ x^2 = \lambda(2y) & \text{—— (2)} \\ x^2+y^2-1 = 0 & \text{—— (3)} \end{cases}$$

1

0.5

- (1) $\Longrightarrow 2xy - 2x\lambda = 0 \Longrightarrow 2x(y-\lambda) = 0 \Longrightarrow \boxed{x = 0 \text{ or } y = \lambda}$

* $x = 0 \overset{(3)}{\Longrightarrow} 0 + y^2 - 1 = 0 \Longrightarrow y^2 = 1 \Longrightarrow y = \pm 1 \Longrightarrow \boxed{(x,y) = (0, \pm 1)}$ 1

(Note: the corresponding λ here is $\lambda = 0$ (from (2)))

** $y = \lambda \overset{(2)}{\Longrightarrow} x^2 = 2y^2 \overset{(3)}{\Longrightarrow} 2y^2 + y^2 - 1 = 0 \Longrightarrow y^2 = \frac{1}{3} \Longrightarrow y = \pm \frac{1}{\sqrt{3}}$ 1

- $y = \frac{1}{\sqrt{3}} \Longrightarrow x^2 = \frac{2}{3} \Longrightarrow x = \pm\sqrt{\frac{2}{3}} \Longrightarrow \boxed{(x,y) = \left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)}$
- $y = -\frac{1}{\sqrt{3}} \Longrightarrow x^2 = \frac{2}{3} \Longrightarrow x = \pm\sqrt{\frac{2}{3}} \Longrightarrow \boxed{(x,y) = \left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right)}$

1

0.5 • $f(0,1) = 0$, $f(0,-1) = 0$, $f\left(\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$, $f\left(-\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$

$f\left(\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$, $f\left(-\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$

0.5 • the max. value of f is $\frac{2}{3\sqrt{3}}$; It occurs at $\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right)$

0.5 The min ———— is $\frac{-2}{3\sqrt{3}}$; ———— $\left(\pm\sqrt{\frac{2}{3}}, \frac{-1}{\sqrt{3}}\right)$

Math201.02, Quiz #3, Term 163

Name:

ID #: **Serial #:**

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1. [6 points] Find the local maximum and minimum values and saddle points of

$$f(x,y) = x^3 - 6xy + 8y^3.$$

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 6y = 0 \\ -6x + 24y^2 = 0 \end{cases} \Rightarrow \begin{cases} x^2 - 2y = 0 & \text{—— (1)} \\ -x + 4y^2 = 0 & \text{—— (2)} \end{cases}$$

$$(2) \Rightarrow x = 4y^2 \overset{(1)}{\Longrightarrow} 16y^4 - 2y = 0 \Rightarrow 2y(8y^3 - 1) = 0 \Rightarrow 2y(2y-1)(4y^2 + 2y + 1) = 0$$

$$\Rightarrow y = 0,\ 1/2,\ \text{Complex}$$

- $y = 0 \Rightarrow x = 0 \Rightarrow (0,0)$
- $y = \frac{1}{2} \Rightarrow x = 1 \Rightarrow (1, \frac{1}{2})$

- $f_{xx}(x,y) = 6x$, $f_{yy}(x,y) = 48y$; $f_{xy}(x,y) = -6$
- $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = 6 \cdot 48 \cdot xy - 36$

* $D(0,0) = -36 < 0 \Rightarrow f$ has a saddle point at $(0,0)$

* $D(1, \frac{1}{2}) = 6 \cdot 48 \cdot \frac{1}{2} - 36 = 3 \cdot 48 - 36 > 0$, $f_{xx}(1, \frac{1}{2}) = 6(1) = 6 > 0$ $\Big\} \Rightarrow f$ has a local min at $(1, \frac{1}{2})$

- $f(0,0) = 0$; $f(1, \frac{1}{2}) = 1 - 3 + 1 = -1$

2. [4 points] Find the volume of the solid below the plane $z = 1 - x$ and above the region bounded by the curves $y = 1 - x^2$ and $y = 0$.

$$V = \iint_R (1-x)\, dA$$

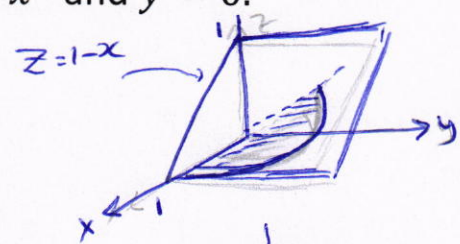
$$= \int_{-1}^{1} \int_{0}^{1-x^2} (1-x)\, dy\, dx$$

$$= \int_{-1}^{1} y - xy \Big]_{y=0}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} (1-x^2) - x(1-x^2) - 0\, dx$$

$$= \int_{-1}^{1} 1 - x^2 - x + x^3\, dx$$

$$= x - \frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x^4 \Big]_{-1}^{1} = \frac{4}{3}$$



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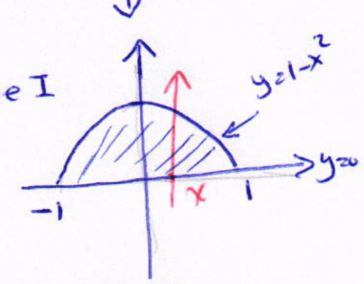

$R: -1 \le x \le 1$
$0 \le y \le 1 - x^2$

3. [5 points] Use Lagrange Multipliers to find the extreme values of $f(x,y) = \frac{1}{x} + \frac{1}{y}$ subject to the constraint $\frac{1}{x^2} + \frac{1}{y^2} = 1$.

- $f(x,y) = \frac{1}{x} + \frac{1}{y}$, $g(x,y) = \frac{1}{x^2} + \frac{1}{y^2} - 1$ $\longleftarrow$ $x \neq 0$, $y \neq 0$
- Solve the System

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Longrightarrow \begin{cases} -x^{-2} = \lambda(-2x^{-3}) & \text{—— (1)} \\ -y^{-2} = \lambda(-2y^{-3}) & \text{—— (2)} \\ \frac{1}{x^2} + \frac{1}{y^2} - 1 = 0 & \text{—— (3)} \end{cases}$$

$$\Longrightarrow \begin{cases} x^3 = 2\lambda x^2 & (1) \\ y^3 = 2\lambda y^2 & (2) \\ \frac{1}{x^2} + \frac{1}{y^2} = 1 & (3) \end{cases}$$

$$\left.\begin{array}{l} (1) \Longrightarrow x = 2\lambda \quad (\text{Since } x \neq 0) \\ (2) \Longrightarrow y = 2\lambda \quad (\text{Since } y \neq 0) \end{array}\right\} \Longrightarrow x = y \text{ —— (4)}$$

$$\xrightarrow{(3)} \frac{1}{x^2} + \frac{1}{x^2} = 1$$

$$\Longrightarrow \frac{2}{x^2} = 1 \Longrightarrow x^2 = 2$$

$$\Longrightarrow x = \pm\sqrt{2}$$

$$\xrightarrow{4} y = \pm\sqrt{2}$$

$$\Longrightarrow \boxed{(x,y) = (\sqrt{2},\sqrt{2}), (-\sqrt{2},-\sqrt{2})}$$

- $f(\sqrt{2},\sqrt{2}) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

  $f(-\sqrt{2},-\sqrt{2}) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

- max. value of f is $\sqrt{2}$; It occurs at $(\sqrt{2},\sqrt{2})$

  min ———— $-\sqrt{2}$; ———— $(-\sqrt{2},-\sqrt{2})$