Master

King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 201- Final Exam – 2016–2017 (163)

Allowed Time: 180 minutes

Name: Key ID #: ____________________

Section #: __________ Serial Number: ______________

Instructions:

1. Make sure that you have **7** questions of written problems and **14** multiple choice questions **(MCQ)**
2. **Show all your work** for written problems. No points for answers without justification.
3. **Calculators, Mobiles and Smart Devices** are not allowed.

Written Problems:

Question #	Grade	Maximum Points
1		12
2		10
3		10
4		10
5		08
6		10
7		10
Total:		70

MCQ Problems:

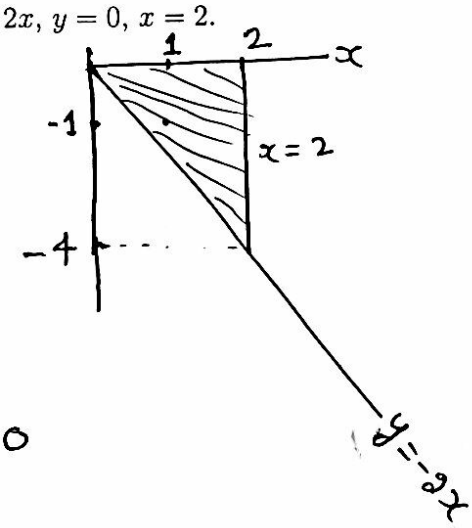
Circle your answer in the table below: 5 points for each correct answer

Question #	Answer		Grade
1	(a)		
2	(a)		
3	(a)		
4	(a)		
5	(a)		
6	(a)		
7	(a)		
8	(a)		
9	(a)		
10	(a)		
11	(a)		
12	(a)		
13	(a)		
14	(a)		
Total:			

1. **[12 points]** Find the absolute maximum and minimum values of

$$f(x,y) = 2x^2 + y^4 + 4xy$$

on the closed triangular region **R** bounded by the lines $y = -2x$, $y = 0$, $x = 2$.



Critical points: $f_x = 4x + 4y = 0 \Rightarrow y = -x$

$f_y = 4y^3 + 4x = 0 \Rightarrow y^3 + x = 0$

$\Rightarrow -x^3 + x = 0$

$\Rightarrow x = 0, \pm 1$

CP $(x,y) = (0,0), (1,-1), (-1,1)$

CP inside the region , $(1,-1)$

On the line: $y = 0$, $0 \le x \le 2$

$f(x,0) = 2x^2 \Rightarrow f'(x,0) = 4x = 0 \Rightarrow x = 0$

Pts $\boxed{(0,0), (2,0)}$

On the line segment: $x = 2$, $-4 \le y \le 0$

$f(2,y) = 8 + y^4 + 8y$

$f'(2,y) = 4y^3 + 8 = 0 \Rightarrow y = (-2)^{1/3}$

Pts $\boxed{(2,(-2)^{1/3}), (2,0), (2,-4)}$

On the line: $y = -2x$, $0 \le x \le 2$

$f(x,-2x) = 2x^2 + 16x^4 - 8x^2 = 16x^4 - 6x^2$

$f'(x,-2x) = 4x(16x^2 - 3) = 0 \Rightarrow x = 0, \pm\frac{\sqrt{3}}{4}$

$\Rightarrow x = 0, \frac{\sqrt{3}}{4} \in [0,2]$

Pts $\boxed{(0,0), (2,-4), \left(\frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{2}\right)}$

f:

$f(0,0) = 0$

$f(2,0) = 8$

$f(2,-4) = 232$ — absalute maximum

$f(1,-1) = -1$ — absalute minimum

$f(2,(-2)^{1/3}) = 8 - 6(2)^{1/3}$

$f\left(\frac{\sqrt{3}}{4}, -\frac{\sqrt{3}}{2}\right) = -\frac{9}{16}$

2. **[10 points]** Use Lagrange multipliers to find maximum and minimum values of the function

$$f(x, y, z) = x - 2y + 5z$$

on the sphere $x^2 + y^2 + z^2 = 30$.

We solve the system

$$\nabla f = \lambda \nabla g \quad \Rightarrow \quad \langle 1, -2, 5 \rangle = \lambda \langle 2x, 2y, 2z \rangle$$

$$g = k \qquad x^2 + y^2 + z^2 = 30 \quad (*)$$

$$\Rightarrow \quad x = \frac{1}{2\lambda}$$

$$y = -\frac{1}{\lambda}$$

$$z = \frac{5}{2\lambda}$$

$(*)$ gives $\left(\frac{1}{2\lambda}\right)^2 + \left(-\frac{1}{\lambda}\right)^2 + \left(\frac{5}{2\lambda}\right)^2 = 30$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{25}{4\lambda^2} = 30$$

$$\Rightarrow 4\lambda^2 = 1$$

$$\Rightarrow \lambda = \pm \frac{1}{2}$$

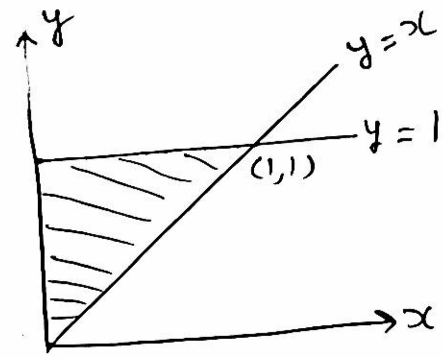
For $\lambda = \frac{1}{2}$, $x = 1$, $y = -2$, $z = 5$

" $\lambda = -\frac{1}{2}$, $x = -1$, $y = 2$, $z = -5$

$f(1, -2, 5) = 30$ max.

$f(-1, 2, -5) = -30$ min

3. **[10 points]** Evaluate $\int_0^1 \int_x^1 e^{x/y}\, dy\, dx$.



$$= \int_0^1 \int_{x=0}^{y} e^{\frac{x}{y}}\, dx\, dy$$

$$= \int_0^1 \left(y\, e^{x/y} \right)_0^y dy$$

Let $\frac{x}{y} = u$

$dx = y\, du$

$$= \int_0^1 (e-1)\, y\, dy$$

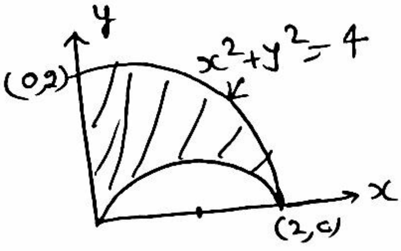
$$= (e-1) \frac{y^2}{2} \Big|_0^1$$

$$= \frac{e-1}{2}$$

4. [10 points] Evaluate $\displaystyle\iint_R y\,dA$, where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

$x^2+y^2 = 4 \Rightarrow r = 2$

$x^2+y^2 = 2x \Rightarrow r = 2\cos\theta$



$$\iint_R y\,dA$$

$$= \int_0^{\pi/2} \int_{2\cos\theta}^{2} r\sin\theta \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(\frac{r^3}{3}\right)_{2\cos\theta}^{2} \cdot \sin\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{3}\left(8 - 8\cos^3\theta\right)\sin\theta\,d\theta$$

$$= \frac{1}{3}\left[-8\cos\theta + \frac{8}{4}\cos^4\theta\right]_0^{\pi/2}$$

$$= \frac{1}{3}(8-2) = \frac{6}{3} = 2.$$

5. [**8 points**] Evaluate $\int_1^4 \int_y^4 \int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy$.

$$= \int_1^4 \int_y^4 z \cdot \frac{1}{z} \tan^{-1}\frac{x}{z}\Big|_0^z \,dz\,dy$$

$$= \int_1^4 \int_y^4 \left(\tan^{-1} 1 - \tan^{-1} 0\right) dz\,dy$$

$$= \frac{\pi}{4} \int_1^4 \int_y^4 dz\,dy$$

$$= \frac{\pi}{4} \int_1^4 (4-y)\,dy$$

$$= \frac{\pi}{4} \left[4y - \frac{y^2}{2}\right]_1^4$$

$$= \frac{\pi}{4} \left[16 - 8 - 4 + \frac{1}{2}\right]$$

$$= \frac{\pi}{4} \left[4 + \frac{1}{2}\right] = \frac{9\pi}{8}$$

6. **[10 points]** Find the volume of the solid bounded by the paraboloids $z = 5 - x^2 - y^2$ and $z = 4x^2 + 4y^2$.

The paraboloids intersect when

$$5 - x^2 - y^2 = 4x^2 + 4y^2$$

$$\Rightarrow x^2 + y^2 = 1$$

$$\text{Volume} = 4\int_0^{\pi/2}\int_0^1\int_{4r^2}^{5-r^2} dZ \cdot r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^1 (5r - r^3 - 4r^3)\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \left(5\frac{r^2}{2} - \frac{5}{4}r^4\right)\Big|_0^1 d\theta$$

$$= 4\int_0^{\pi/2} \frac{5}{4}\,d\theta$$

$$= \frac{5\pi}{2}$$

or $$\int_0^{2\pi}\int_0^1\int_{4r^2}^{5-r^2} dZ \cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{5}{4}\,d\theta$$

$$= \frac{5\pi}{2}$$

7. **[10 points]** Use spherical coordinates to find the volume of the part of the ball $x^2 + y^2 + z^2 \le 4$ that lies between the cones

$$2z = \sqrt{2x^2 + 2y^2} \text{ and } 2z = \sqrt{x^2 + y^2}.$$

In spherical the cones are given by

$z = \frac{1}{2}\sqrt{2x^2+2y^2}$

$\frac{z}{r} = \frac{\sqrt{2}}{2}$

$\cot\phi = \frac{1}{\sqrt{2}}$

$\tan\phi = \sqrt{2}$

$\Rightarrow \phi = \tan^{-1}(\sqrt{2})$

$z = \frac{1}{2}\sqrt{x^2+y^2}$

$\tan\phi = 2$

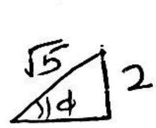
$\phi = \tan^{-1}(2)$

$$V = \int_0^{2\pi} \int_{\tan^{-1}(\sqrt{2})}^{\tan^{-1}(2)} \int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2\pi \left[\frac{1}{3}\rho^3\right]_0^2 \left[-\cos\phi\right]_{\tan^{-1}(\sqrt{2})}^{\tan^{-1}(2)}$$

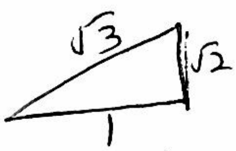
$$= 2\pi \cdot \frac{8}{3}\left[-\cos\tan^{-1}(2) + \cos\tan^{-1}(\sqrt{2})\right]$$

$$= \frac{16\pi}{3}\left[+\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}}\right]$$



$\cos\phi = \frac{1}{\sqrt{5}}$

$\phi = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$



$\cos\phi = \frac{1}{\sqrt{3}}$

1. The Cartesian equation of the curve defined by

$$x = \cos(\pi - t),\ y = \sin(\pi - t),\ 0 \le t \le \pi$$

is

t	x	y
0	-1	0
$\frac{\pi}{2}$	0	1
π	1	0

(a) $x^2 + y^2 = 1, \quad -1 \le x \le 1, \quad 0 \le y \le 1$

(b) $x^2 + y^2 = 1, \quad 0 \le x \le 1, \quad 0 \le y \le 1$

(c) $x^2 + y^2 = 1, \quad -1 \le x \le 1, \quad -1 \le y \le 1$

(d) $x^2 - y^2 = 1, \quad -1 \le x \le 1, \quad 0 \le y \le 1$

(e) $x^2 + y^2 = \pi^2, \quad 0 \le x \le 1, \quad 0 \le y \le 1$

2. The graph of $r = 4 \tan\theta \sec\theta$ is

$$r = 4\,\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$

$$= 4\,\frac{y}{r}\cdot\frac{r}{x}\cdot\frac{r}{x}$$

$$y = \frac{x^2}{4}.$$

(a) a parabola

(b) a circle

(c) an ellipse

(d) a straight line

(e) a hyperbola

3. Length of the polar curve

$$r = \sqrt{1 + \sin 2\theta},\ 0 \le \theta \le \sqrt{2}\,\pi$$

is equal to:

(a) 2π

(b) 1

(c) π

(d) 4π

(e) 4

$$\frac{dr}{d\theta} = \frac{\cos 2\theta}{\sqrt{1+\sin 2\theta}}$$

$$s = \int_0^{\sqrt{2}\pi} \sqrt{1 + \sin 2\theta + \frac{\cos^2 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$= \int_0^{\sqrt{2}\pi} \sqrt{\frac{1 + \sin^2 2\theta + 2\sin 2\theta + \cos^2 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$= \int_0^{\sqrt{2}\pi} \sqrt{\frac{1 + 1 + 2\sin 2\theta}{1+\sin 2\theta}}\, d\theta$$

$$= \int_0^{\sqrt{2}\pi} \sqrt{2}\, d\theta = \sqrt{2}\cdot\sqrt{2}\,\pi$$

$$= 2\pi$$

4. The quadratic surface $x^2 - y^2 + z^2 - 2x + 2y - 4z + 2 = 0$ represents

$$(x-1)^2 - (y-1)^2 + (z-2)^2 = 2$$

(a) a hyperboloid of one sheet

(b) a hyperboloid of two sheets

(c) an ellipsoid

(d) a cone

(e) a paraboloid

5. The value of k for which the vectors $\vec{a} = < 4, -3, 1 >$, $\vec{b} = < 2, 1, 3 >$ and $\vec{c} = < k, 0, 2 >$ lie in the same plane is

(a) $k = 2$

(b) $k = 1$

(c) $k = -1$

(d) $k = -2$

(e) $k = 0$

$$\begin{vmatrix} 4 & -3 & 1 \\ 2 & 1 & 3 \\ k & 0 & 2 \end{vmatrix} = 0$$

$$8 + 3(4 - 3k) - k = 0$$
$$20 - 9k - k = 0$$
$$20 - 10k = 0$$
$$k = 2$$

6. If the angle between two unit vectors $\vec{u}_1$ and $\vec{u}_2$ is $\frac{\pi}{4}$, then $|2\vec{u}_1 - \vec{u}_2|^2 =$

(a) $5 - 2\sqrt{2}$

(b) $3 - 2\sqrt{2}$

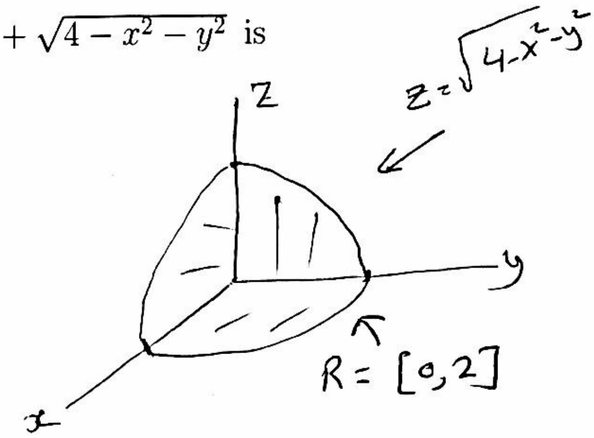
(c) $1 + 2\sqrt{2}$

(d) $5 + 2\sqrt{2}$

(e) $3 + 2\sqrt{2}$

$$|2\vec{u}_1 - \vec{u}_2|^2 = (2\vec{u}_1 - \vec{u}_2) \cdot (2\vec{u}_1 - \vec{u}_2)$$
$$= 4|\vec{u}_1|^2 - 4\vec{u}_1 \cdot \vec{u}_2 + |\vec{u}_2|^2$$
$$= 4 - 4|\vec{u}_1||\vec{u}_2| \cos\frac{\pi}{4} + 1$$
$$= 5 - 4\frac{\sqrt{2}}{2}$$
$$= 5 - 2\sqrt{2}$$

7. The range of the function $F(x, y) = 1 + \sqrt{4 - x^2 - y^2}$ is



(a) $[1, 3]$

(b) $[0, 2]$

(c) $[-1, 0]$

(d) $[2, 5]$

(e) $[1, \infty)$

8. If $L = \lim\limits_{(x,y)\to(0,0)} \dfrac{e^{1+x^2+y^2} - e}{x^2 + y^2}$, then

(a) $L = e$

(b) L does not exist

(c) $L = 0$

(d) $L = 1$

(e) $L = 2e$

Put $r^2 = x^2 + y^2$

$$L = \lim_{r \to 0^+} \frac{e^{1+r^2} - e}{r^2}$$

$$= \lim_{r \to 0^+} \frac{2r\, e^{1+r^2}}{2r}$$

$$= \lim_{r \to 0^+} e^{1+r^2}$$

$$= e$$

9. Let $F(x,y,z) = x^2\sqrt{y+z} + \dfrac{2y}{xz}$.

The value of $F_{xy} + F_{xz}$ at the point $(-1,0,1)$ is

(a) -4

(b) -3

(c) -1

(d) 2

(e) 0

$$F_x = 2x\sqrt{y+z} - \frac{2y}{x^2 z}$$

$$F_{xy} = \frac{2x}{2\sqrt{y+z}} - \frac{2}{x^2 z}$$

$$F_{xz} = \frac{2x}{2\sqrt{y+z}} + \frac{2y}{x^2 z^2}$$

$$F_{xy} + F_{xz}\Big|_{(-1,0,1)} = -1 - 2 - 1 + 0 = -4$$

10. Let $z = xy^2 + x^2y,\ x = r + s\cos t,\ y = s + r\sin t$.

The value of $\dfrac{\partial z}{\partial t}$, when $r = 2$, $s = 1$, $t = \pi$, is equal to

(a) -6

(b) -3

(c) 1

(d) 2

(e) 5

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

$$= (y^2 + 2xy)(-s\sin t) + (2xy + x^2)(r\cos t)$$

$$\frac{\partial z}{\partial t}\Big|_{(2,1,\pi)} = 0 + (2+1)(2\cos\pi) = (3)(-2) = -6$$

$$x = 2 + 1\cos\pi = 2 - 1 = 1$$

$$y = 1 + 2\sin\pi = 1 + 0 = 1$$

11. The directional derivative of $f(x, y, z) = xe^y + ye^z + ze^x$ at the point $(1, 1, 1)$ in the direction of the vector $\vec{v} = < 5, 1, -2 >$ is equal to

(a) $\frac{8e}{\sqrt{30}}$

(b) $\frac{2e}{\sqrt{30}}$

(c) $\frac{e}{\sqrt{30}}$

(d) $\frac{-3e}{\sqrt{30}}$

(e) $\frac{-4e}{\sqrt{30}}$

$\nabla f = \langle e^y + ze^x, xe^y + e^z, ye^z + e^x \rangle$

$\nabla f(1,1,1) = \langle e+e, e+e, e+e \rangle$

$= 2e \langle 1,1,1 \rangle$

$\nabla f \cdot \hat{u} = 2e \langle 1,1,1 \rangle \cdot \frac{\langle 5,1,-2 \rangle}{\sqrt{25+1+4}}$

$= \frac{2e(5+1-2)}{\sqrt{30}} = \frac{8e}{\sqrt{30}}$

12. The function $f(x, y) = 1+2x+4y-x^2-2y^2$ has one critical point (a, b) and

(a) f has a local maximum at (a, b)

(b) f has a local minimum at (a, b)

(c) f has a saddle point at (a, b)

(d) $f(a, b) = 8$

(e) $f_{xy}(a, b) = 3$

$f_x = 2 - 2x = 0 \Rightarrow x = 1$

$f_y = 4 - 4y = 0 \Rightarrow y = 1$

$f_{xx} = -2$

$f_{xy} = 0$

$f_{yy} = -4$

$D = 8 > 0, \ f_{xx} < 0$

13. The value of the double integral

$$\int_0^2 \int_0^{y^2} 3\,y^3\,e^{xy}\,dx\,dy$$

is equal to:

(a) $e^8 - 9$

(b) $e - 1$

(c) $e^6 - 5$

(d) $e^4 - 5$

(e) $e^8 - 8$

$xy = u$

$dx = \frac{du}{y}$

$$\int_0^2 \left[3y^3 \frac{e^{xy}}{y} \right]_0^{y^2} dy$$

$$= \int_0^2 \left(3y^2 e^{y^3} - 3y^2 \right) dy$$

$$= \left[e^{y^3} - y^3 \right]_0^2$$

$$= e^8 - 8 - (e^0 - 0)$$

$$= e^8 - 9$$

14. If D is the triangular region with vertices $(0,1)$, $(1,2)$, $(4,1)$, then $\iint_D y\,dA =$

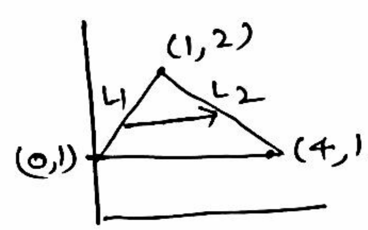
(a) $\frac{8}{3}$

(b) $\frac{5}{3}$

(c) $\frac{1}{3}$

(d) $\frac{10}{3}$

(e) $\frac{2}{3}$



L_1: $y - 1 = \frac{2-1}{1-0}(x - 0)$

$y - 1 = x$

L_2: $y - 2 = \frac{1-2}{4-1}(x - 1)$

$y - 2 = -\frac{1}{3}(x - 1)$

$3y - 6 = 1 - x \Rightarrow x = 7 - 3y$

$$\iint_D y\,dA = \int_{y=1}^{2} \int_{y-1}^{7-3y} y\,dx\,dy$$

$$= \int_1^2 y\left[7 - 3y - y + 1\right] dy = \int_1^2 y(8 - 4y)\,dy$$

$$= 8\frac{y^2}{2} - 4\frac{y^3}{3} \Big|_1^2$$

$$= \frac{32}{2} - \frac{32}{3} - \frac{8}{2} + \frac{4}{3}$$

$$= \frac{24}{2} - \frac{28}{3} = 12 - \frac{28}{3} = \frac{36 - 28}{3} = \frac{8}{3}$$

Q	MM	V1	V2	V3	V4
1	a	a	a	b	c
2	a	a	e	e	c
3	a	d	b	a	c
4	a	d	d	c	b
5	a	d	a	b	c
6	a	b	e	b	d
7	a	a	e	e	c
8	a	c	d	e	e
9	a	d	e	b	a
10	a	a	c	d	c
11	a	d	a	c	b
12	a	c	c	c	d
13	a	e	c	d	b
14	a	d	b	e	c