Izhar

King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 201 Major Exam I
The Third Semester of 2016-2017 (163)

Time Allowed: 120 Minutes

Name: ______________ ID#: ______________

Section/Instructor: key Serial #: ______________

- Mobiles, calculators and smart devices are not allowed in this exam.
- Provide all necessary steps required in the solution.

Question #	Marks	Maximum Marks
1		12
2		13
3		10
4		11
5		16
6		12
7		14
8		12
Total		100

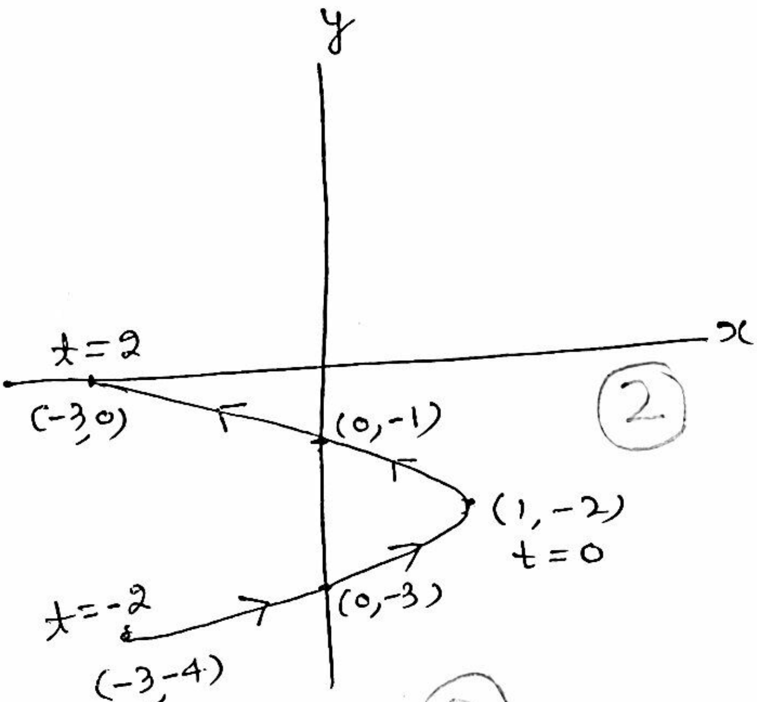
Q:1 (8 + 4 points) Consider the parametric equations of a curve C :

$$x = 1 - t^2, \quad y = t - 2, \qquad -2 \le t \le 2.$$

(a) Sketch the curve and indicate with an arrow the direction in which the curve is traced as t increases.

t	x	y
-2	-3	-4
-1	0	-3
0	1	-2
1	0	-1
2	-3	0

4



(b) Eliminate t to find the corresponding cartesian equation.

$y = t - 2 \Rightarrow t = y + 2,$ 1

so $x = 1 - t^2$

$= 1 - (y+2)^2, \qquad -4 \le y \le 0$ 1 + 2

Q:2 (5 + 3 + 5 points) Consider the parametric equations of a curve C :

$$x = t^3 - 12\,t, \quad y = t^2 - 1.$$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{3t^2-12} \quad ① + ①$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad ①$$

$$= \frac{2(3t^2-12) - 2t\cdot 6t/(3t^2-12)^2}{(3t^2-12)} \quad ①$$

$$= \frac{-6t^2-24}{(3t^2-12)^3}$$

$$= \frac{-6\,(t^2+4)}{3^3\,(t^2-4)^3} \quad ①$$

(b) For which values of t is the curve concave upward ?

The curve is CU when $\frac{d^2y}{dx^2} > 0$ ①

The curve is CU when $t^2 - 4 < 0$ ①

$\Rightarrow |t| < 2$

$\Rightarrow -2 < t < 2$ ①

(c) Find an equation of the tangent line to the curve C when $t = 1$.

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{2}{3-12} = -\frac{2}{9} \quad ①$$

$$(x,y) = (1-12, 1-1) = (-11, 0). \quad ②$$

Equation of tangent line is

$$y - 0 = -\frac{2}{9}(x+11) \quad ②$$

or

$$9y = -2x - 22.$$

Q:3 (10 points) Find the length of the curve defined by

$$x = t\cos(t), \quad y = t\sin(t), \; 0 \le t \le 1.$$

$$\text{Length} = \int_{t=\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad 2$$

$$\frac{dx}{dt} = \cos t - t\sin t \qquad 1$$

$$\frac{dy}{dt} = \sin t + t\cos t \qquad 1$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t$$

$$= 1 + t^2 \qquad 2$$

$$\text{Length} = \int_0^1 \sqrt{1+t^2}\, dt$$

Let $t = \tan\theta$
$dt = \sec^2\theta\, d\theta$

$$s = \int_0^{\pi/4} \sec\theta \cdot (1+\tan^2\theta)\, d\theta \qquad 1$$

$$= \int_0^{\pi/4} \sec\theta\, d\theta + \int_0^{\pi/4} \underbrace{\tan\theta}_{u} \cdot \underbrace{\sec\theta\tan\theta\, d\theta}_{dv} \qquad 1$$

$$= \ln(\sec\theta + \tan\theta)\Big|_0^{\pi/4} + \tan\theta\sec\theta\Big|_0^{\pi/4} - \int_0^{\pi/4} \sec\theta \cdot \sec^2\theta\, d\theta \qquad 1$$

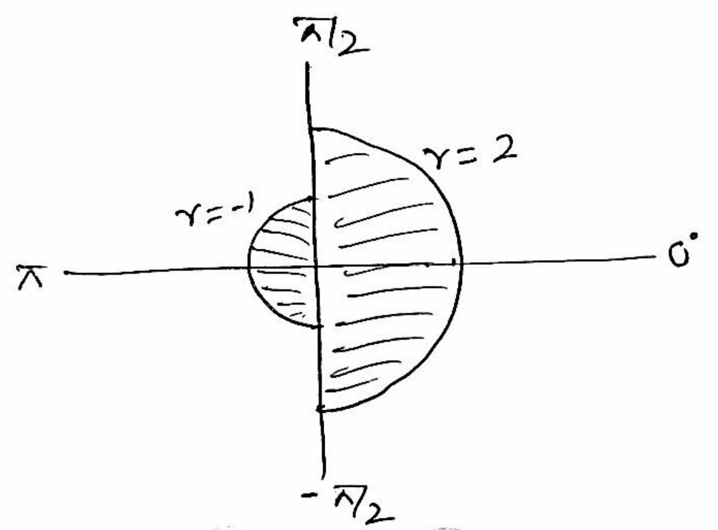
$$2s = \ln(\sqrt{2}+1) + \sqrt{2} \qquad 1$$

$$s = \frac{1}{2}\ln(\sqrt{2}+1) + \frac{\sqrt{2}}{2}.$$

Q:4 (a)(3 points) Find polar coordinate (r, θ) of the rectangular point $(-2, 2)$ in such a way that $r < 0$ and $2\pi \le \theta < 4\pi$.

$$r^2 = x^2 + y^2$$
$$= 2^2 + 2^2$$
$$= 4 + 4 = 8 \quad (1)$$
$$r = -2\sqrt{2} \quad (r < 0) \quad (1)$$
$$\theta = 2\pi + \left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{11\pi}{4} \quad (1)$$
$$(r, \theta) = \left(-2\sqrt{2}, \frac{11\pi}{4}\right)$$

(b)(4 points) Sketch the region in the plane consisting of points whose polar coordinates satisfy the conditions $-1 \le r \le 2$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

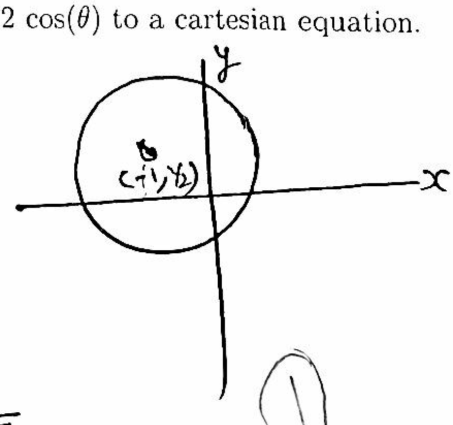


(2) + (2)

(c) (4 points) Convert the polar equation $r = \sin(\theta) - 2\cos(\theta)$ to a cartesian equation. Sketch the resulting equation.

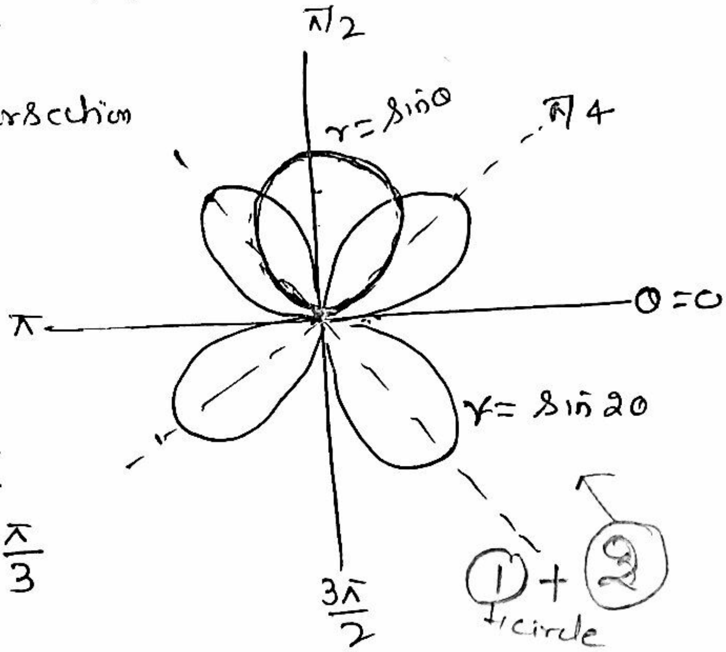
$$r = \sin\theta - 2\cos\theta$$
$$= \frac{y}{r} - 2\frac{x}{r} \quad (1)$$
$$\Rightarrow r^2 = y - 2x$$
$$\Rightarrow x^2 + y^2 + 2x - y = 0 \quad (1)$$
$$\Rightarrow (x+1)^2 + \left(y - \frac{1}{2}\right)^2 = \frac{5}{4} \quad (1)$$

Circle with center $(-1, 1/2)$ and radius $\sqrt{\frac{5}{4}}$



(1)

Q:5 (a)(8 points) Sketch on the same polar coordinate system the curves
$$r = \sin(\theta) \text{ and } r = \sin(2\theta)$$
and find the point(s) of intersection if any.



1 The pole is a point of intersection

$\sin\theta = \sin 2\theta$

$= 2\sin\theta\cos\theta$

1 $\sin\theta(1-2\cos\theta)=0$

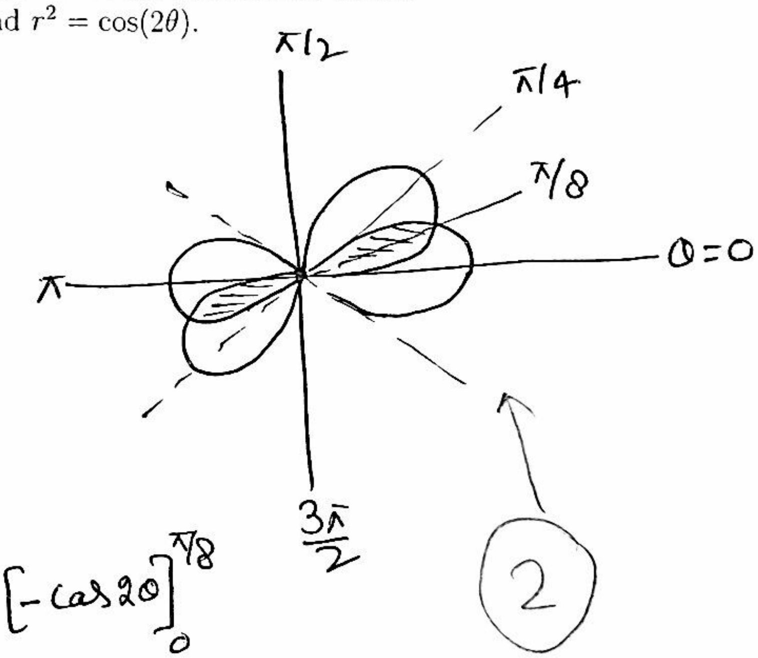
$\Rightarrow \sin\theta = 0$ or $\cos\theta = \frac{1}{2}$

1 $\Rightarrow \theta = 0, \pi, \frac{\pi}{3}$ or $-\frac{\pi}{3}$

The other points of intersection are

2 $\left(\frac{\sqrt{3}}{2}, \frac{\pi}{3}\right)$ and $\left(\frac{\sqrt{3}}{2}, \frac{2\pi}{3}\right)$

(b) (8 points) Find the area of the region that lies inside both curves
$$r^2 = \sin(2\theta) \text{ and } r^2 = \cos(2\theta).$$



Points of intersection are

$\sin 2\theta = \cos\theta$

$\tan 2\theta = 1$

$2\theta = \frac{\pi}{4} + 2n\pi$

2 $\theta = \frac{\pi}{8}$ or $\frac{9\pi}{8}$

2 $A = 4\int_0^{\pi/8} \frac{1}{2}\sin 2\theta\, d\theta$

$= \int_0^{\pi/8} 2\sin 2\theta\, d\theta = \left[-\cos 2\theta\right]_0^{\pi/8}$

2 $= -\frac{\sqrt{2}}{2} + 1$

Q:6 (a)(4 points) Find an equation of the sphere that passes through the point $(0, 1, -1)$ and whose center is $(2, 0, 1)$.

$$r = \sqrt{(2-0)^2 + (0-1)^2 + (1+1)^2}$$

$$= \sqrt{4+1+4} = 3 \quad 2$$

An equation of the sphere is

$$(x-2)^2 + y^2 + (z-1)^2 = 9 \quad 2$$

(b) (3 points) Write inequalities to describe the region between (but not on) the yz-plane and the plane $x = 8$.

$$0 < x < 8 \quad 2+1$$

(c) (5 points) Let $A(-2, 0, 1)$ and $B(1, 3, -2)$ be two points in three dimensional space.

(i) Find $\overrightarrow{AB}$.

(ii) Find a unit vector that has the same direction of $\overrightarrow{AB}$.

(iii) Find a vector of magnitude 7 and having the opposite direction of $\overrightarrow{AB}$.

(i) $\overrightarrow{AB} = \langle 1+2, 3-0, -2-1 \rangle = \langle 3, 3, -3 \rangle$ 1

(ii) Unit vector $= \dfrac{\langle 3, 3, -3 \rangle}{\sqrt{27}} = \dfrac{3 \langle 1, 1, -1 \rangle}{3\sqrt{3}}$ 2

(iii) Unit vector in the direction of $\overrightarrow{AB}$ is $\hat{v}$ $= \dfrac{\langle 1, 1, -1 \rangle}{\sqrt{3}}$

A vector in the opposite direction but magnitude 7 is $-7\vec{v} = \dfrac{\langle -7, -7, 7 \rangle}{\sqrt{3}}$ 2

Q:7 Let $\vec{a} =< 3, 0, -1 >$, $\vec{b} =< -1, 4, 1 >$ and $\vec{c} =< 12, 1, 2 >$ be three vectors.

(a) (4 points) Find the scalar projection of $\vec{b}$ onto $\vec{c}$.

Scalar projection $\vec{b}$ onto $\vec{C}$ $= \dfrac{\vec{b} \cdot \vec{C}}{|\vec{C}|}$ 2

$= \dfrac{-12 + 4 + 2}{\sqrt{144 + 4 + 1}} = \dfrac{-6}{\sqrt{149}}$ 2

(b) (5 points) Find the vector projection of $\vec{b}$ onto $\vec{c}$.

Vector projection $\vec{b}$ onto $\vec{C}$ $= \dfrac{\vec{b} \cdot \vec{C}}{|\vec{C}|^2} \vec{C}$ 2

$= \dfrac{-6}{149} \langle 12, 1, 2 \rangle$ 2

$= \dfrac{\langle -72, -6, -12 \rangle}{149}$ 1

(c) (5 points) Find two unit vectors that are orthogonal to both $\vec{a} =< 3, 0, -1 >$ and $\vec{b} =< -1, 4, 1 >$.

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 3 & 0 & -1 \\ -1 & 4 & 1 \end{vmatrix}$$

$= \langle 4, -2, 12 \rangle$ 2

$|\vec{A} \times \vec{B}| = \sqrt{16 + 4 + 144} = \sqrt{164}$

$= 2\sqrt{41}$ 1

Unit vectors $= \dfrac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \pm \dfrac{\langle 2, -1, 6 \rangle}{\sqrt{41}}$ 2

Q:8 (12 points) Find the volume of the parallelepiped determined by the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, and $\overrightarrow{OC}$ where O is the origin and $A(1,2,-1)$, $B(-2,0,3)$, $C(0,7,-4)$.

$\overrightarrow{OA} = \langle 1, 2, -1 \rangle$ 2

$\overrightarrow{OB} = \langle -2, 0, 3 \rangle$ 2

$\overrightarrow{OC} = \langle 0, 7, -4 \rangle$ 2

2 $\overrightarrow{OA} \cdot (\overrightarrow{OB} \times \overrightarrow{OC}) = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$ 2

$= 1(0-21) - 2(8-0) - 1(-14-0)$

$= -21 - 16 + 14$

$= -23$ 1

Volume $= |-23| = 23 \text{ unit}^3$. 1