1. The slope of the tangent line to the curve

$$y = \frac{1}{x+3}$$

at $x = 0$ equals

(a) $-\frac{1}{9}$
(b) $\frac{1}{9}$
(c) 1
(d) -1
(e) 0

$m = f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0} \frac{\frac{1}{x+3}-\frac{1}{3}}{x}$

$= \lim_{x\to 0} \frac{3-(x+3)}{x(x+3)(3)}$

$= \lim_{x\to 0} \frac{-\cancel{x}}{\cancel{x}(x+3)(3)}$

$= \frac{-1}{9}$

2. If $[[x]]$ is the greatest integer less than or equal to x, then

$$\lim_{x\to 0^+} [[\tan(-x)]] =$$

(a) -1
(b) 1
(c) 0
(d) $-\frac{\pi}{2}$
(e) $-\pi$

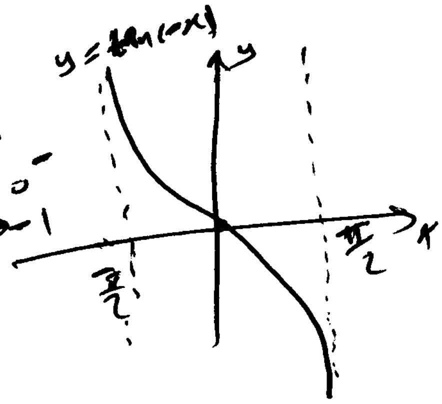
$x \to 0^+$

$-x \to 0^-$

$\tan(-x) < 0$

$\tan(-x) \to 0^-$

$[[\tan(-x)]] \to -1$



3. $\lim\limits_{x\to 0^-} \left(\frac{1}{|x|} + \frac{1}{x}\right) =$

$$\lim_{x\to 0^-}\left(\frac{1}{-x}+\frac{1}{x}\right) = \lim_{x\to 0^-} 0 = 0.$$

(a) 0

(b) ∞

(c) $-\infty$

(d) 1

(e) $\frac{1}{2}$

4. $\lim\limits_{x\to 1} \frac{\sqrt{x}-x}{1-\sqrt{x}} =$

$$\lim_{x\to 1}\frac{(\sqrt{x}-x)(\sqrt{x}+x)(1+\sqrt{x})}{(1-\sqrt{x})(1+\sqrt{x})(\sqrt{x}+x)}$$

$$= \lim_{x\to 1}\frac{(x-x^2)(1+\sqrt{x})}{(1-x)(\sqrt{x}+x)}$$

$$= \lim_{x\to 1}\frac{x\cancel{(1-x)}(1+\sqrt{x})}{\cancel{(1-x)}(\sqrt{x}+x)}$$

$$= \frac{(1)(2)}{(2)} = 1.$$

(a) 1

(b) $\frac{1}{2}$

(c) 0

(d) ∞

(e) $-\infty$

5. Which of the following functions

$$f(x) = \frac{x^2-1}{(x-1)^2},\ g(x) = \ln(\frac{x^2-2x+1}{x-1}),$$
$$h(x) = \frac{x^2+x-2}{x-1},\ p(x) = \tan^{-1}(\frac{1}{|x-1|})$$

has a vertical asymptote at $x = 1$?

(a) $f(x)$ and $g(x)$ only
(b) $f(x)$, $g(x)$ and $h(x)$ only
(c) $g(x)$ and $h(x)$ only
(d) $f(x)$, $g(x)$, $h(x)$ and $p(x)$
(e) $f(x)$ and $h(x)$ only

• $\lim_{x\to1^+}\frac{x^2-1}{(x-1)(x-1)} = \lim_{x\to1^+}\frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(x-1)} = \infty$ V.a.

• $\lim_{x\to1^+}\ln\left(\frac{x^2-2x+1}{x-1}\right) = \lim_{x\to1^+}\ln\left(\frac{\cancel{(x-1)}(x-1)}{\cancel{x-1}}\right) = \lim_{u\to0^+}\ln u = -\infty$ V.a.

$\boxed{u = x-1}$

• $\lim_{x\to1}\frac{x^2+x-2}{x-1} = \lim_{x\to1}\frac{(x+2)\cancel{(x-1)}}{\cancel{x-1}} = 3$ No V.a

• $\lim_{x\to}\tan^{-1}\left(\frac{1}{|x-1|}\right) = \tan^{-1}\left(\lim_{u\to0}\frac{1}{|u|}\right) = \frac{\pi}{2}$. No V.a

$u = x-1$

6. Given that $\lim_{x\to1}(2-3x) = -1$, and using the ϵ,δ-definition, the largest possible value of δ that corresponds to $\epsilon = 0.06$ is

(a) 0.02
(b) 0.01
(c) 0.03
(d) 0.04
(e) 0.06

Given $\epsilon>0$, $\exists\ \delta>0$ such that

$0<|x-1|<\delta \Rightarrow |2-3x-(-1)|<\epsilon$

$|3-3x|<\epsilon$

$3|1-x|<\epsilon$

$3|x-1|<\epsilon$

$|x-1|<\frac{\epsilon}{3}$

$0<\delta\le\frac{\epsilon}{3}$

$\epsilon = 0.06$

$0<\delta\le\frac{0.06}{3}$

$0<\delta\le 0.02$

7. The function $f(x) = \dfrac{x^2 - 3x + 2}{x^2 + x - 6}$ has

(a) two discontinuities: one removable and one infinite $f(x) = \dfrac{(x-2)(x-1)}{(x-2)(x+3)}$

(b) two removable discontinuities

(c) two infinite discontinuities $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+3)} = \dfrac{1}{5}$

(d) only one discontinuity which is removable — removable discontinuity at $x = 2$.

(e) only one discontinuity which is infinite

$$\lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} \frac{\cancel{(x-2)}(x-1)}{\cancel{(x-2)}(x+3)} = -\infty$$

$f(x)$ has an infinite disc. at $x = -3$.

8. $\displaystyle\lim_{x\to\infty} \frac{x - 2\sin(3x)}{5x+1} =$

(a) $\dfrac{1}{5}$

(b) 1

(c) -1

(d) ∞

(e) $-\infty$

$$\lim_{x\to\infty} \frac{\frac{x}{x} - 2\frac{\sin 3x}{x}}{\frac{5x}{x} + \frac{1}{x}} = \frac{1-0}{5+0} = \frac{1}{5}$$

$$-1 \le \sin 3x \le 1$$

$$-\frac{1}{x} \le \frac{\sin 3x}{x} \le \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{-1}{x} = 0 = \lim_{x\to\infty} \frac{1}{x}$$

So $\displaystyle\lim_{x\to\infty} \frac{\sin 3x}{x} = 0$ by the Squeezing theorem

9. The function $f(x) = \dfrac{\sqrt{x^2+x-12}}{x-3}$ is continuous on

(a) $(-\infty, -4] \cup (3, \infty)$

(b) $(-\infty, 3) \cup [3, \infty)$

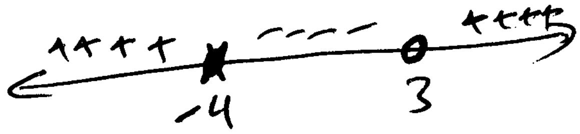
(c) $(-\infty, 3) \cup (3, \infty)$

(d) $(-\infty, -4) \cup (-4, 3) \cup (3, \infty)$

(e) $(-\infty, \infty)$

$x^2+x-12 \geq 0$ & $x \neq 3$.

$(x+4)(x-3) \geq 0$ & $x \neq 3$



$x \in (-\infty, -4] \cup (3, \infty)$

10. $\lim\limits_{x \to -\infty} (\sqrt{x^2+3x} - x) =$

$\lim\limits_{x\to-\infty} \sqrt{x(x+3)} - \lim\limits_{x\to-\infty} x$

$\sqrt{(-\infty)(-\infty)} + \infty$

$\infty + \infty = \infty$.

(a) ∞

(b) $-\infty$

(c) $\dfrac{3}{2}$

(d) $-\dfrac{3}{2}$

(e) 0

11. The graph of the function $f(x) = \dfrac{\sqrt{x^2+2}}{2x-1}$ has

(a) one vertical asymptote and two horizontal asymptotes

(b) one vertical asymptote and only one horizontal asymptote

(c) one vertical asymptote and no horizontal asymptotes

(d) no vertical asymptotes and one horizontal asymptote

(e) no vertical asymptotes and two horizontal asymptotes

- $\lim_{x\to\frac{1}{2}^+} f(x) = \lim_{x\to\frac{1}{2}^+} \dfrac{\sqrt{x^2+2}}{2x-1} \to \dfrac{\frac{3}{2}}{0^+} = \infty$ vertical asymptote at $x = \frac{1}{2}$
- $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{\sqrt{x^2(1+\frac{2}{x^2})}}{2x-1} = \lim_{x\to\infty} \dfrac{\cancel{x}\sqrt{1+\frac{2}{x^2}}}{\cancel{x}(2-\frac{1}{x})} = \dfrac{1}{2-0} = \dfrac{1}{2}$

$\therefore$ $y = \frac{1}{2}$ is a horizontal asymptote

- $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \dfrac{\sqrt{x^2(1+\frac{2}{x^2})}}{2x-1} = \lim_{x\to-\infty} \dfrac{|x|\sqrt{1+\frac{2}{x^2}}}{x(2-\frac{1}{x})} = \lim_{x\to-\infty} \dfrac{-\cancel{x}\cdot\sqrt{1+\frac{2}{x^2}}}{\cancel{x}(2-\frac{1}{x})} = \dfrac{-\sqrt{1+0}}{2} = -\dfrac{1}{2}$

$y = -\frac{1}{2}$ is a horizontal asymptote.

12. If the function $f(x) = \begin{cases} 3x-2, & x<0 \\ ax^2+b, & 0 \le x \le 1 \\ \dfrac{x^2-1}{x-1}, & x>1 \end{cases}$ is continuous everywhere, then $a - b =$

(a) 6

(b) 4

(c) 2

(d) 1

(e) 0

- $\lim_{x\to0^-} f(x) = \lim_{x\to0^+} f(x)$

$\lim_{x\to0^-}(3x-2) = \lim_{x\to0^+}(ax^2+b)$

$\boxed{-2 = b}$

- $\lim_{x\to1^-} f(x) = \lim_{x\to1^+} f(x)$

$\lim_{x\to1^-}(ax^2+b) = \lim_{x\to1^+} \dfrac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$

$\boxed{a+b = 2}$

$\boxed{\therefore a = 2-b = 4}$

- $a-b = 4-(-2) = 6$

13. If the tangent of $y = f(x)$ at the point $(1, 2)$ passes through the point $(2, 3)$, then $f'(1) =$

(a) 1

(b) 0

(c) 2

(d) 3

(e) 4

$$f'(1) = m_{\text{tangent}} = \frac{\Delta y}{\Delta x} = \frac{3-2}{2-1} = \frac{1}{1} = 1$$

14. The vertical tangent of $f(x) = (1 - 2x)^{1/5}$ is

(a) $x = \frac{1}{2}$

(b) $x = 2$

(c) $x = -2$

(d) $x = \frac{-1}{2}$

(e) $x = 0$

Solution I

Notice that $1 - 2x = 0 \Rightarrow x = \frac{1}{2}$

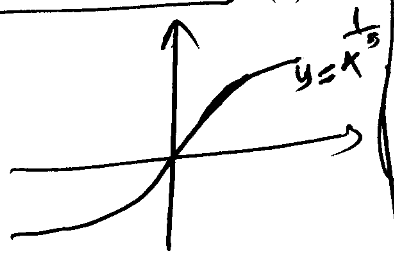
$$\lim_{x\to\frac{1}{2}} |f'(x)| = \lim_{x\to\frac{1}{2}} \left|\frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}}\right| = \lim_{x\to\frac{1}{2}} \left|\frac{(1-2x)^{\frac{1}{5}} - 0}{x - \frac{1}{2}}\right| = \lim_{u\to 0} \left|\frac{(-2u)^{\frac{1}{5}} - 0}{u}\right| = 2^{\frac{1}{5}} \cdot \lim_{u\to 0} \left|\frac{u^{\frac{1}{5}}}{u^1}\right| = 2^{\frac{1}{5}} \lim_{u\to 0} \frac{1}{u^{\frac{4}{5}}} = \infty$$

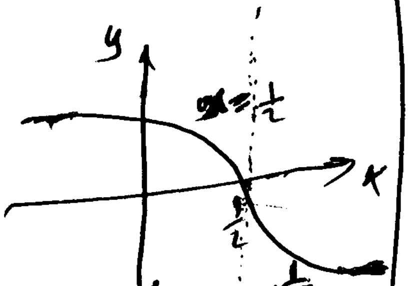
$$\boxed{\text{Let } u = x - \tfrac{1}{2} \quad x \to \tfrac{1}{2},\ u \to 0}$$

$$1 - 2x = 1 - 2(u + \tfrac{1}{2}) = 1 - 2u + 1 = -2u$$

$\therefore x = \frac{1}{2}$ is a vertical tangent.

Solution II





$y = (1 - 2x)^{\frac{1}{5}}$

15. If $f(x) = x^2 - 2x + 3$, then $f'(2) =$

(a) 2

(b) 0

(c) 1

(d) 3

(e) 4

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - 2(2+h) + 3 - 3}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4} + 4h + h^2 \cancel{- 4} - 2h + 0}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(4 + h - 2)}{\cancel{h}}$$

$$= 2.$$

16. If $f(x) = \begin{cases} 3, & x \leq 2 \\ 2x - 9, & 2 < x < 5 \\ \dfrac{1}{2x-9}, & x \geq 5 \end{cases}$

then $f'_-(5) + f'_+(5) =$

(a) 0

(b) 1

(c) 2

(d) -1

(e) -2

- $f(5) = \frac{1}{2(5) - 9} = 1.$

- $f'_-(5) = \lim_{h \to 0^-} \frac{f(5+h) - f(5)}{h}$

$$= \lim_{x \to 5^-} \frac{f(x) - f(5)}{x - 5}$$

$$= \lim_{x \to 5^-} \frac{(2x-9) - 1}{x - 5}$$

$$= \lim \frac{2\cancel{(x-5)}}{\cancel{x-5}} = 2.$$

- $f'_+(5) = \lim_{x \to 5^+} \frac{f(x) - f(5)}{x - 5}$

$$= \lim_{x \to 5^+} \frac{\frac{1}{2x-9} - 1}{x - 5}$$

$$= \lim_{x \to 5^+} \frac{1 - (2x-9)}{(2x-9)(x-5)}$$

$$= \lim_{x \to 5^+} \frac{10 - 2x}{(2x-9)(x-5)}$$

$$= \lim_{x \to 5^+} \frac{-2\cancel{(x-5)}}{(2x-9)\cancel{(x-5)}}$$

$$= -2$$

17. A particle moves along a straight line with equation of motion

$$s = f(t) = t^{-1} - t,\ t > 0$$

where s is measured in meters and t is measured in seconds. The **speed** when $t = 5$ is

$f(5) = \frac{1}{5} - \frac{5}{1} = \frac{-24}{5}$

(a) $\frac{26}{25}\, m/s$

(b) $-\frac{26}{25}\, m/s$

(c) $\frac{25}{26}\, m/s$

(d) $-\frac{25}{26}\, m/s$

(e) $-\frac{5}{6}\, m/s$

$$V(5) = \lim_{t \to 5} \frac{f(t) - f(5)}{t-5} = \lim_{t\to 5} \frac{\frac{1}{t} - \frac{t}{1} + \frac{24}{5}}{t-5} = \lim_{t \to 5} \frac{5 - 5t^2 + 24t}{5t(t-5)} = \lim_{t\to 5} - \frac{(5t^2 - 24t - 5)}{5t(t-5)} = \lim_{t\to 5} - \frac{(5t+1)\cancel{(t-5)}}{5t\cancel{(t-5)}} = -\frac{26}{25}$$

Speed $= |V(5)|$

Speed$\Big|_{t=5} = \left|\frac{-26}{25}\right| = \frac{26}{25}$

18. Let

$$f(x) = \begin{cases} |x+1|, & x \le 2 \\ \dfrac{-1}{x-3}, & 2 < x < 5 \\ \dfrac{1}{4}x + 5, & x \ge 5 \end{cases}$$

The number of points at which $f(x)$ is not differentiable is

(a) 4

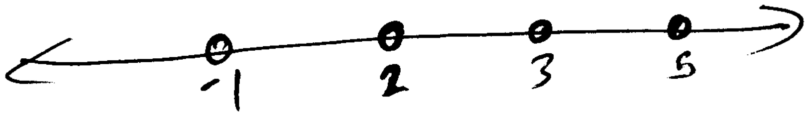
(b) 0

(c) 1

(d) 3

(e) 2

$\boxed{\text{notice that not cts. at } a \Downarrow \text{ not diff. at } a.}$



- $f(x)$ is (cts. but) not diff at $x = -1$
- $f(x)$ not cts. at $x = 2$, since $\lim_{x\to 2^-} f(x) = \lim_{x \to 2^-} |x+1| = 3$, while $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \frac{-1}{x-3} = \frac{-1}{-1} = 1$.
- $f(x)$ not cts. at $x = 3$
- $f(x)$ not cts. at $x = 5$; $\lim_{x\to 5^-} f(x) = \frac{-1}{2} \neq \frac{25}{4} = \lim_{x \to 5^+} f(x)$

19. Let

$$f(x) = \begin{cases} \dfrac{1}{x^2-1}, & x \le 0 \\ x-5, & x > 0 \end{cases}$$

Using the graph of $g(x)$, we conclude that $\lim_{x\to 0}(f-g)(x)$

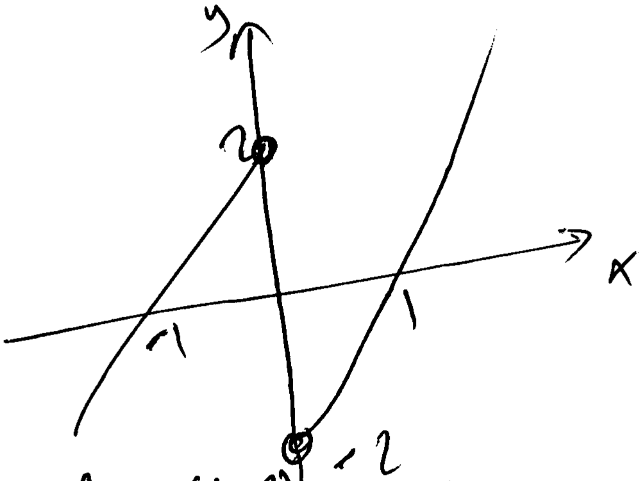
(a) equals -3

(b) equals 1

(c) equals 0

(d) equals 2

(e) does not exists



$\lim_{x\to 0^-}(f-g)(x) = \lim_{x\to 0^-} f - \lim_{x\to 0^-} g = (-1-2) = -3$

$\lim_{x\to 0^+}(f-g)(x) = \lim_{x\to 0^+} f - \lim_{x\to 0^+} g = -5-(-2) = -3.$

$\therefore \lim_{x\to 0}(f-g)(x) = -3$

20. If $[[x]]$ is the greatest integer less than or equal to x, then $\lim_{x\to -1^+}\left[\left[\dfrac{1}{2-x^2}\right]\right] =$

(a) 0

(b) 1

(c) -1

(d) ∞

(e) $-\infty$

when $x \to -1^+$ we have

$x > -1$

$x^2 < 1$

$-x^2 > -1$

$2 - x^2 > 1$

$\dfrac{1}{2-x^2} < 1$

$\left[\left[\dfrac{1}{2-x^2}\right]\right] \longrightarrow 0.$