# STAT460 Time Series Formula

## CHAPTER 2 – Fundamental Concepts

| **Stochastic Process Model** | **Mean** $E(Y_t)$ | **Autocovariance function,** $\gamma_{t,s} = Cov(Y_t, Y_s)$ | **Autocorrelation function,** $\rho_{t,s}$ |
|---|---|---|---|
| $\{Y_t : t = 0, \pm 1, \pm 2, \pm 3, \ldots\}$ | $\mu_t = E(Y_t)$ (2.2.1) | $\gamma_{t,s} = Cov(Y_t, Y_s)$ (2.2.2) | $\rho_{t,s} = Corr(Y_t, Y_s)$ (2.2.3) $= \frac{Cov(Y_t, Y_s)}{\sqrt{Var(Y_t)Var(Y_s)}} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}}$ (2.2.4) |
| $Y_t = Y_{t-1} + e_t$ (2.2.9) with $Y_1 = e_1$ "initial condition" | $\mu_t = 0$ (2.2.10) for all $t$ | $\gamma_{t,s} = t\sigma_e^2$ (2.2.12) for $1 \le t \le s$ <br> $Var(Y_t) = t\sigma_e^2$ (2.2.11) | $\rho_{t,s} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} = \sqrt{\frac{t}{s}}$ (2.2.13) for $1 \le t \le s$ |
| $Y_t = \frac{e_t + e_{t-1}}{2}$ (2.2.14) | $\mu_t = 0$ for all $t$ | $\gamma_{t,t-s} = \begin{cases} 0.5\sigma_e^2 & \text{for } \lvert t-s\rvert = 0 \\ 0.25\sigma_e^2 & \text{for } \lvert t-s\rvert = 1 \\ 0 & \text{for } \lvert t-s\rvert > 1. \end{cases}$ (2.2.15) | $\rho_{t,t-s} = \begin{cases} 1 & \text{for } \lvert t-s\rvert = 0 \\ 0.5 & \text{for } \lvert t-s\rvert = 1 \\ 0 & \text{for } \lvert t-s\rvert > 1 \end{cases}$ (2.2.16) |
| Random Cosine Wave $Y_t = cos(2\pi(\frac{t}{12} + \Phi))$ for $t = 0, \pm 1, \pm 2, \ldots$ | $\mu_t = 0$ for all $t$ | $\gamma_{t,s} = \frac{1}{2} cos(2\pi(\frac{\lvert t-s\rvert}{12}))$. | $\rho_{t,s} = cos(2\pi \frac{k}{12})$ (2.3.4) for $k = 0, \pm 1, \pm 2, \ldots$ |

$$\gamma_{t,t} = Var(Y_t) \quad \gamma_{t,s} = \gamma_{s,t} \quad \lvert\gamma_{t,s}\rvert \le \gamma_{t,t}\gamma_{s,s} \quad (2.2.5)$$
$$\rho_{t,t} = 1 \quad \rho_{t,s} = \rho_{s,t} \quad \lvert\rho_{t,s}\rvert \le 1$$

$Cov(\sum_{i=1}^{m} c_i Y_{t_i}, \sum_{j=1}^{n} d_j Y_{s_j}) = \sum_{i=1}^{m}\sum_{j=1}^{n} c_i d_j Cov(Y_{t_i}, Y_{s_j})$ with constants $c_i$ and $d_j$ (2.2.6)

Special case: $Var(\sum_{i=1}^{n} c_i Y_{t_i}) = \sum_{i=1}^{n} c_i^2 Cov(Y_{t_i}) + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} c_i c_j Cov(Y_{t_i}, Y_{t_j})$ (2.2.7)

**strictly stationary** process $\{Y_t\}$ if the joint distribution of $Y_{t1}, Y_{t2}, \ldots, Y_{tn}$ is the same as the joint distribution of $Y_{t1-k}, Y_{t2-k}, \ldots, Y_{tn-k}$ for all choices of time points $t_1, t_2, \ldots, t_n$ and all choices of time lag $k$.

**weakly**(or **second-order**) **stationary** if

1) The mean function is constant over time, and
2) $\gamma_{t,t-k} = \gamma_{0,k}$ for all time $t$ and lag $k$.

# CHAPTER 3–TRENDS

Residual $\hat{X}_t = Y_t - \hat{\mu}_t$ (3.6.1) $\qquad r_k = \frac{\sum_{t=k+1}^{n}(Y_t-\bar{Y})(Y_{t-k}-\bar{Y})}{\sum_{t=1}^{n}(Y_t-\bar{Y})^2}$ for $k = 1, 2, \cdots$ (3.6.2)

| | | |
|---|---|---|
| Most stationary processes | $Var(\bar{Y}) \approx \frac{\gamma_0}{n}\left[\sum_{k=-\infty}^{\infty}\rho_k\right]$ for large $n$ (3.2.5) | $\sum_{k=0}^{n-1}\lvert\rho_k\rvert \le \infty$ (3.2.4) Except random cosine wave |
| Special stationary | $Var(\bar{Y}) \approx \frac{(1+\phi)}{(1-\phi)}\frac{\gamma_0}{n}$ (3.2.6) | $\rho_k = \phi^{\lvert k\rvert}$ for all $k$, where $-1<\phi<1$ |

| **Model** | **Estimate of Mean** $E(Y_t)$ | $\boldsymbol{Var(\bar{Y})}$ | **Autocorrelation** |
|---|---|---|---|
| $Y_t = \mu + X_t$, (3.2.1)<br>*where* $E(X_t) = 0$ | $\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t$ (3.2.2) | $Var(\bar{Y}) = \frac{\gamma_0}{n}\left[\sum_{k=-n+1}^{n-1}\left(1-\frac{\lvert k\rvert}{n}\right)\rho_k\right]$<br>$= \frac{\gamma_0}{n}\left[1 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\rho_k\right]$ (3.2.3) | |
| $Y_t = \mu + X_t$, *where* $E(X_t) = 0$<br>$\{X_t\}$ white noise | $\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t$ | $Var(\bar{Y}) = \frac{\gamma_0}{n}$ | |
| $Y_t = e_t - \frac{1}{2} e_{t-1}$ MA *model* | $\bar{Y} = \frac{1}{n}\sum_{t=1}^{n} Y_t$ | $Var(\bar{Y}) = \frac{\gamma_0}{n}\left[1 - 0.8\left(\frac{n-1}{n}\right)\right]$<br>$Var(\bar{Y}) \approx 0.2\frac{\gamma_0}{n}$ for large $n$ | $\rho_1 = -0.4$<br>$\rho_k = 0$ for $k > 1$ |
| $Y_t = X_t$, random walk<br>*where* $E(X_t) = 0$ $X_t = \sum_{i=1}^{t} e_i$ | | $Var(\bar{Y}) = \frac{1}{n^2}\left(\sigma_e^2\sum_{t=1}^{n} t^2\right)$<br>$= (2n+1)\frac{(n+1)}{6n}\sigma_e^2$ (3.2.7) | |
| $Y_t = \mu_t + X_t$,<br>*where* $E(X_t) = 0$<br>$\mu_t = \beta_0 + \beta_1 t$<br>(3.3.1) | $\hat{\beta}_1 = \frac{\sum_{t=1}^{n}(Y_t-\bar{Y})(t-\bar{t})}{\sum_{t=1}^{n}(t-\bar{t})^2}$ $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{t}$ (3.3.2)<br>*where*<br>$\bar{t} = \frac{1}{n}\sum_{t=1}^{n} t = \frac{n+1}{2}$<br>Alternatively (3.4.7)<br>$\hat{\beta}_1 = \frac{\sum_{t=1}^{n}(t-\bar{t})Y_t}{\sum_{t=1}^{n}(t-\bar{t})^2}$ | | |
| as above but<br>$\mu_t = \mu_{t-12}$<br>$\mu_t = \beta_j$ $j = \mathrm{mod}(t,12)$<br>$j = 12$ if $\mathrm{mod}(t,12) = 0$ | Dummy variable regression estimates<br>$\hat{\beta}_j = \frac{1}{N}\sum_{i=0}^{N-1} Y_{j+12i}$ | $Var(\hat{\beta}_j) = \frac{\gamma_0}{N}\left[1 + 2\sum_{k=1}^{N-1}\left(1-\frac{k}{N}\right)\rho_{12k}\right]$<br>for $j = 1, 2, \ldots, 12$ (3.4.1) | |
| $Y_t = \mu_t + X_t$,<br>*where* $E(X_t) = 0$<br>$\mu_t = \beta\cos(2\pi ft + \Phi)$ (3.3.4)<br>$\mu_t = \beta_1\cos(2\pi ft) + \beta_2\sin(2\pi ft)$ (3.3.5)<br>where $\beta = \sqrt{\beta_1^2 + \beta_2^2}$<br>$\Phi = \mathrm{atan}(-\beta_2/\beta_1)$ (3.3.6) conversely $\beta_1 = \beta\cos(\Phi)$<br>$\beta_2 = -\beta\sin(\Phi)$ (3.3.7) | $\hat{\beta}_1 = \frac{2}{n}\sum_{t=1}^{n}\left[\cos\left(\frac{2\pi mt}{n}\right)Y_t\right]$ $\hat{\beta}_2 = \frac{2}{n}\sum_{t=1}^{n}\left[\sin\left(\frac{2\pi mt}{n}\right)Y_t\right]$<br>(3.4.2) | $Var(\hat{\beta}_1) = \frac{2\gamma_0}{N}\left[1 + 4\sum_{s=2}^{n}\sum_{t=1}^{s-1}\cos\left(\frac{2\pi mt}{n}\right)\cos\left(\frac{2\pi ms}{n}\right)\rho_{s-t}\right]$<br>(3.4.3) | |
| Same as above random cosine but<br>$\{X_t\}$ is white noise | As above | $Var(\hat{\beta}_1) = \frac{2\gamma_0}{N}\left[1 + 4\frac{\rho_1}{n}\sum_{t=1}^{n-1}\cos\left(\frac{\pi t}{6}\right)\cos\left(\frac{\pi t+1}{6}\right)\right]$ (3.4.4)<br>$Var(\hat{\mu}_1) = Var(\hat{\beta}_0) + Var(\hat{\beta}_1)\left[\cos\left(\frac{2\pi}{12}\right)\right]^2 + Var(\hat{\beta}_2)\left[\sin\left(\frac{2\pi}{12}\right)\right]^2$<br>(3.4.6) | $\rho_1 \neq 0$,<br>$\rho_k = 0$<br>for $k > 1$ &<br>$m/n = 1/12$ |

| $n$ | 25 | 50 | 500 | $\infty$ |
|---|---|---|---|---|
| $Var(\hat{\beta}_1)$ | $\frac{2\gamma_0}{n}(1 + 1.66\rho_1)$ | $\frac{2\gamma_0}{n}(1 + 1.70\rho_1)$ | $\frac{2\gamma_0}{n}(1 + 1.73\rho_1)$ | $\frac{2\gamma_0}{n}\left(1 + 2\rho_1\cos\left(\frac{\pi}{6}\right)\right)$<br>$= \frac{2\gamma_0}{n}(1 + 1.732\rho_1)$ (3.4.5) |

## CHAPTER 4 - MODELS FOR STATIONARY TIME SERIES

***General linear process,*** $\{Y_t\}$, $Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \ldots$ (4.1.1) assume $\sum_{j=1}^{\infty} \psi_j^2 < \infty$ (4.1.2)

Case $\psi_j = \phi^j$ where $-1 < \phi < 1$ $Corr(Y_t, Y_{t-k}) = \phi^k$ (4.1.3)

$E(Y_t) = 0$ $\gamma_k = Cov(Y_t, Y_{t-k}) = \sigma_e^2 \sum_{i=1}^{\infty} \psi_i \psi_{i+k} k \geq 0$ (4.1.4) with $\psi_0 = 1$.

**MA Process:** $Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}$ (4.2.1)

$\gamma_0 = Var(Y_t) = \sigma_e^2 (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)$ (4.2.4)

$$\rho_1 = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & \text{for } k = 1, 2, \cdots, q \\ 0 & \text{for } k > q \end{cases}$$

**MA(1) model:** $Y_t = e_t - \theta_1 e_{t-1}$, $E(Y_t) = 0$, $\gamma_0 = Var(Y_t) = \sigma_e^2 (1 + \theta^2)$,

$\gamma_1 = -\theta\sigma_e^2$, $\rho_1 = (-\theta)/(1 + \theta^2)$, $\gamma_k = \rho_k = 0$ for $k \geq 2$ (4.2.2)

**MA(2) model:** $Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$ $\gamma_0 = Var(Y_t) = Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) = Var(Y_t) = \sigma_e^2 (1 + \theta_1^2 + \theta_2^2)$

$$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) = Cov(-\theta_1 e_{t-1}, e_{t-1}) + Cov(-\theta_1 e_{t-2}, -\theta_2 e_{t-2})$$
$$= [-\theta_1 + (-\theta_1)(-\theta_2)]\sigma_e^2 = (-\theta_1 + \theta_1\theta_2)\sigma_e^2.$$

$$\gamma_2 = Cov(Y_t, Y_{t-2}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) = Cov(-\theta_2 e_{t-2}, e_{t-2}) = -\theta_2\sigma_e^2.$$

**First-Order Autoregressive AR(1) Process:** Assuming $|\phi| < 1$, $Y_t = \phi_1 Y_{t-1} + e_t$ (4.3.2)

$\gamma_0 = \frac{\sigma_e^2}{1-\phi^2}$ (4.3.3) $\gamma_k = \phi\gamma_{k-1}$ (4.3.4) $\gamma_k = \phi^k \gamma_0 = \phi^k \frac{\sigma_e^2}{1-\phi^2}$ (4.3.5) $\rho_k = \gamma_k/\gamma_0 = \phi^k$ (4.3.6)

***General Linear Process Version:*** Assuming $|\phi| < 1$,**:** $Y_t = e_t + \phi e_{t-1} + \phi^2 Y_{t-2} + \ldots + \phi^{k-1} Y_{t-k+1} + \phi^k Y_{t-k}$ (4.3.7)

$Y_t = e_t + \phi e_{t-1} + \phi^2 Y_{t-2} + \phi^3 Y_{t-3} + \ldots$ (4.3.8)

**AR(2) model:** $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$ (4.3.9)

**AR characteristic polynomial:** $\phi(x) = 1 - \phi_1 x - \phi_2 x^2$

**AR characteristic equation:** $\phi(x) = 0 \rightarrow 1 - \phi_1 x - \phi_2 x^2 = 0$ $\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$ (4.3.10)

**Stationarity of the AR(2):** $\phi_1 + \phi_2 < 1$, $\phi_2 - \phi_1 < 1$, and $|\phi_2| < 1$ (4.3.11)

**Autocorrelation Function for the AR(2):** $\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2}$ for $k = 1, 2, 3, \ldots$ (4.3.12)

**Yule-Walker equations:** $\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$ for $k = 1, 2, 3, \ldots$ (4.3.13)

$\rho_1 = \frac{\phi_1}{1-\phi_2}$ (4.3.14) $\rho_2 = \frac{\phi_2(1-\phi_2) + \phi_1^2}{1-\phi_2}$ (4.3.15) $G_1 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$ $G_2 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$

$\rho_k = \frac{(1-G_2^2)G_1^{k+1} - (1-G_1^2)G_2^{k+1}}{(G_1 - G_2)(1 + G_1 G_2)}$ for $k \geq 0$ (4.3.16)

$\rho_k = R^k \frac{Sin\,(\Theta k + \Phi)}{Sin\,(\Phi)}$ for $k \geq 0$ (4.3.17) for complex roots where $R = \sqrt{-\phi_2}$

$\cos(\Theta) = \phi_1/(-2\sqrt{-\phi_2})$ and $\tan(\Phi) = (1 - \phi_2)/(1 + \phi_2)$

$\rho_k = \left(1 + \frac{1-\phi_2}{1+\phi_2}k\right)\left(\frac{\phi_1}{2}\right)^k$ for same roots for $k$ = 0, 1, 2,...

**Variance for the AR(2):** $\gamma_0 = (\phi_1^2 + \phi_2^2)\gamma_0 + 2\phi_1\phi_2\gamma_1 + \sigma_e^2$ (4.3.19)

$\gamma_0 = \frac{(1-\phi_2)\sigma_e^2}{(1-\phi_2)(1-\phi_1^2-\phi_2^2)-2\phi_2\phi_1^2} = \frac{(1-\phi_2)}{(1+\phi_2)(1-\phi_2)^2-\phi_1^2}\sigma_e^2$ (4.3.20) $\gamma_1 = \phi_1\gamma_0 + \phi_2\gamma_1$

**ψ-Coefficients for the AR(2):** $\psi_0 = 1$, $\psi_1 - \phi_1\psi_0 = 0$, $\psi_j - \phi_1\psi_{j-1} - \phi 2\psi_{j-2} = 0$ $j = 2, 3, \ldots$ (4.3.21)

$\psi_j = \frac{G_1^{j+1} - G_2^{j+1}}{(G_1 - G_2)}$ (4.3.22) $\psi_j = R^j \frac{Sin\,([j+1]\Theta)}{Sin\,(\Theta)}$ (4.3.23) for complex roots

$\psi_j = (1+j)(\phi_1/2)^j$ (4.3.24) for same roots

**General Autoregressive AR($p$) Process:** $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t$ (4.3.25)

AR characteristic polynomial: $\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p$ (4.3.26)

AR characteristic equation: $1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p = 0.$ (4.3.27)

Stationarity conditions: $\phi_1 + \phi_2 + \ldots + \phi_p < 1$ and $|\phi_j| < 1$ (4.3.28)

Recursive Relationship: $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \ldots + \phi_p\rho_{k-p}$ for $k \geq 1$ (4.3.29)

**Yule-Walker equations:**

$$\left.\begin{aligned} \rho_1 &= \phi_1 + \phi_2\rho_1 + \phi_3\rho_2 + \ldots + \phi_p\rho_{p-1} \\ \rho_2 &= \phi_1\rho_1 + \phi_2 + \phi_3\rho_1 + \ldots + \phi_p\rho_{p-2} \\ &\vdots \\ \rho_p &= \phi_1\rho_{p-1} + \phi_2\rho_{p-2} + \phi_3\rho_{p-3} + \ldots + \phi_p \end{aligned}\right\} \quad (4.3.30)$$

$\gamma_0 = \frac{\sigma_e^2}{1-\phi_1\rho_1 - \phi_2\rho_2 - \ldots - \phi_p\rho_p}$ (4.3.31)

**ARMA($p$,$q$):** $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}$ (4.4.1)

**ARMA(1,1):** $Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$ (4.4.2)

$\gamma_0 = \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2$ $\gamma_1 = \phi\gamma_0 - \theta\sigma_e^2$ $\gamma_k = \phi\gamma_{k-1}$ for $k \geq 2$ (4.4.3)

$\gamma_0 = \left(\frac{1-2\theta\phi+\theta^2}{1-\phi^2}\right)\sigma_e^2$ (4.4.4) $\rho_k = \frac{(1-\theta\phi)(\phi-\theta)}{1-2\theta\phi+\theta^2}\phi^{k-1}$ for $k \geq 1$ (4.4.5)

**General linear process ARMA(1,1) version:** $Y_t = e_t + (\phi - \theta)\sum_{j=1}^{\infty}\phi^{j-1}e_{t-j}$, (4.4.6)

$\psi_j = (\phi - \theta)\phi^{j-1}$ for $j \geq 1$

**ARMA($p$,$q$)** general linear process with ψ-coefficients determined from

$\psi_0 = 1$, $\psi_1 = -\theta_1 + \phi_1$, $\psi_2 = -\theta_2 + \phi_2 + \phi_1\psi_1$, $\ldots\psi_j = -\theta_j + \phi_p\psi_{j-p} + \phi_{p-1}\psi_{j-p+1} + \ldots + \phi_1\psi_{j-1}$ (4.4.7)

the autocorrelation function: $\rho_k = \phi_1\rho_{k-1} + \phi_2\rho_{k-2} + \ldots + \phi_p\rho_{k-p}$ for $k > q$. (4.4.8)

**MA characteristic polynomial:** $\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \theta_3 x^3 - \ldots - \theta_q x^q$ (4.5.3)

**MA characteristic equation** $\theta(x) = 0$: $1 - \theta_1 x - \theta_2 x^2 - \theta_3 x^3 - \ldots - \theta_q x^q = 0$ (4.5.4)

MA($q$) model is **invertible**; with coefficients $\pi_j$ so that $Y_t = \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \pi_3 Y_{t-3} + \ldots + e_t$ (4.5.5)

## CHAPTER 5 - MODELS FOR NONSTATIONARY TIME SERIES

| **Model** | **Mean** E($Y_t$) | **Autocovariance function,** $\gamma_{t,s} = Cov\,(Y_t, Y_s)$ | **Autocorrelation function,** $\rho_{t,s}$ |
|---|---|---|---|
| $Y_t = \phi Y_{t-1} - e_t$ (5.1.1) | | | |
| $Y_t = 3Y_{t-1} - e_t$ (5.1.2)<br>$Y_t = e_t + 3e_{t-1} + 3^2 e_{t-2} + \ldots$ $+ 3^{t-1}e_1 + 3^t Y_0$ (5.1.3) | | $Var(Y_t) = \frac{1}{8}(9^t - 1)\sigma_e^2$ (5.1.4)<br>$Cov(Y_t, Y_{t-1}) = \frac{3^k}{8}(9^{t-k} - 1)\sigma_e^2$ (5.1.5) | $Corr(Y_t, Y_{t-1}) = 3^k\sqrt{\frac{9^{t-k}-1}{9^t-1}}$ for large $t$ and moderate $k$. |
| $Y_t = M_t + e_t$ (5.1.9) with $M_t = M_{t-1} + \varepsilon_t$ | | | $\rho_1 = -\{1/[2 + (\sigma_\varepsilon^2/\sigma_e^2)]$ (5.1.10) |

| | | | |
|---|---|---|---|
| where $\{e_t\}$ and $\{\varepsilon_t\}$ are independent white noise series | | | |
| $Y_t = Y_{t-1} - e_t$ (5.1.6) | | | |
| $Y_t = M_t + e_t$ *with*<br>$M_t = M_{t-1} + W_t$ *and*<br>$W_t = W_{t-1} + \varepsilon_t$ (5.1.11) | | | $\nabla^2 Y_t$ has acf of an MA(2) |
| ARIMA($p$,1,$q$) | | | |
| IMA(1,1) | | $Var(Y_t) = [1 + \theta^2 + (1-\theta)^2(t+m)]\sigma_e^2$ | $Corr(Y_t, Y_{t-k}) = \dfrac{1 - \theta + \theta^2 + (1-\theta)^2(t+m-k)}{[Var(Y_t)Var(Y_{t-k})]^{1/2}}$<br>$\approx \sqrt{\dfrac{t+m-k}{t+m}}$<br>$\approx 1$ for large $m$ and moderate $k$ |
| IMA(2,2)<br>$Y_t = 2Y_{t-1} - Y_{t-2} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$ | | | |
| ARI(1,1)<br>$Y_t - Y_{t-1} = \phi(Y_{t-1} - Y_{t-2}) + e_t$ | | | |

ARI(1,1): $-(1+\phi) + \psi_1 = 0$, $\phi - (1+\phi)\psi_1 + \psi_2 = 0$ $\psi_0 = 1$ and $\psi_1 = 1 + \phi$. $\psi_k = (1+\phi)\psi_{k-1} - \phi\psi_{k-2}$ for $k \ge 2$ $\psi_k = \dfrac{1-\phi^{k+1}}{1-\phi}$ for $k \ge 1$

ARIMA($p$,$d$,$q$)

$$(1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi^p x^p)(1-x)^d(1 + \psi_1 x + \psi_2 x^2 + \psi_3 x^3 + \cdots)$$
$$= (1 - \theta_1 x - \theta_2 x^2 - \theta_3 x^3 - \cdots - \theta_q x^q)$$

| **Transformation** | | **Effects** |
|---|---|---|
| *differencing* | $\nabla^d Y_t = W_t$ | Reduce to ARMA($p$,$q$) model |
| *Logarithmic* | If $Y_t > 0$ for all $t$ and<br>$E(Y_t) = \mu_t$ and $\sqrt{Var(Y_t)} = \mu_t\sigma$<br>$\log(Y_t) \approx \log(\mu_t) + \dfrac{Y_t - \mu_t}{\mu_t}$ | $E[\log(Y_t)] \approx \log(\mu_t)$ and $Var(\log(Y_t)) \approx \sigma^2$ |
| *Difference in log* | $\log(Y_t) - \log(Y_{t-1}) = \log\left(\dfrac{Y_t}{Y_{t-1}}\right) = \log(1 + X_t)$<br>If $\lvert X_t \rvert < 0.2$, $\nabla[\log(Y_t)] \approx X_t$ | |
| *Power*<br>*or Box-Cox* | $g(x) = \begin{cases} \dfrac{x^\lambda - 1}{\lambda} & \text{for } \lambda \ne 0 \\ \log x & \text{for } \lambda = 0 \end{cases}$<br>$\lambda = ½ =$ square root transformation<br>$\lambda = -1 =$ reciprocal transformation<br>Typical $\lambda = 0, \pm 1, \pm 1/2, \pm 1/3$, or $\pm 1/4$ | |

## CHAP 6 MODEL SPECIFICATIONS

$r_k = \dfrac{\sum_{t=k+1}^{n}(Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{n}(Y_t - \bar{Y})^2}$ for $k = 1, 2, \cdots$ (6.1.1)

If $Y_t = \mu + \sum_{j=0}^{\infty}\psi_j e_{t-j}$ ($e_t$ are i.i.d zero means and finite, non-zero common variances). Also, $\sum_{j=0}^{\infty}\lvert\psi_j\rvert < \infty$ and $\sum_{j=0}^{\infty} j\psi_j^2 < \infty$ (True for any stationary ARMA model)

Then, for any fixed $m$, the joint distribution of $\sqrt{n}(r_1 - \rho_1)$, $\sqrt{n}(r_2 - \rho_2), \cdots, \sqrt{n}(r_m - \rho_m)$, as $n \to \infty$, approaches a joint normal distribution with zero means, variances $c_{jj}$, and covariances $c_{ij}$

$c_{ij} = \sum_{k=-\infty}^{\infty}(\rho_{k+i}\rho_{k+j} + \rho_{k-i}\rho_{k+j} - 2\rho_i\rho_k\rho_{k+j} - 2\rho_j\rho_k\rho_{k+i} + 2\rho_i\rho_j\rho_k^2)$ (6.1.2)

$Corr(r_k, r_j) \approx c_{kj}/\sqrt{c_{kk}c_{jj}}$ $Var(r_k) \approx \frac{1}{n}$ and $Corr(r_k, r_j) \approx 0$ for $k \neq j$ (6.1.3)

If $\{Y_t\}$ follows AR(1) process with $\rho_k = \Phi^k$ for $k > 0$, $Var(r_k) \approx \frac{1}{n}\left[\frac{(1+\Phi^2)(1-\Phi^{2k})}{1-\Phi^2} - 2k\Phi^{2k}\right]$ (6.1.4)

$Var(r_1) \approx \frac{1-\Phi^2}{n}$ (6.1.5) Approximate $Var(r_k) \approx \frac{1}{n}\left[\frac{1+\phi^2}{1-\phi^2}\right]$ for large $k$ (6.1.6)

For the AR(1) model, $0 < i < j$ as $c_{ij} = \frac{(\phi^{j-i}-\phi^{j+1})(1+\phi^2)}{1-\phi^2} + (j-i)\phi^{j-i} - (j+i)\phi^{j+i}$ (6.1.7)

$Corr(r_1, r_2) \approx 2\phi\sqrt{\frac{1-\phi^2}{1+2\phi^2-3\phi^4}}$ (6.1.8)

For the MA(1) case, $c_{11} = 1 - 3\rho_1^2 + 4\rho_1^4$ and $c_{kk} = 1 + 2\rho_1^2$ for $k > 1$ (6.1.9) $c_{12} = 2\rho_1(1-\rho_1^2)$ (6.1.10)

For the MA($q$) process, $c_{kk} = 1 + 2\sum_{j=1}^{q}\rho_j^2$ for $k > q$ and

$Var(r_k) = \frac{1}{n}\left[1 + 2\sum_{j=1}^{q}\rho_j^2\right]$ $for\ k > q$ (6.1.11) $\phi_{kk} = Corr(Y_t, Y_{t-k}|Y_{t-1}, Y_{t-2}, \cdots, Y_{t-k+1})$ (6.2.1)

$\phi_{kk} = Corr(Y_t - \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + \cdots + \beta_{k-1}Y_{t-k+1}, Y_{t-k} - \beta_1 Y_{t-k+1} + \beta_2 Y_{t-k+2} + \cdots + \beta_{k-1}Y_{t-1})$ (6.2.2)

By convention, we take $\phi_{11} = 1$. $Cov(Y_t - \rho_1 Y_{t-1}, Y_{t-1} - \rho_1 Y_{t-2}) = \gamma_0(\rho_2 - \rho_1^2 + \rho_1^2 - \rho_1^2) = \gamma_0(\rho_2 - \rho_1^2)$

Since $Var(Y_t - \rho_1 Y_{t-1}) = Var(Y_{t-1} - \rho_1 Y_{t-2}) = \gamma_0(1 + \rho_1^2 - 2\rho_1^2) = \gamma_0(1-\rho_1^2)$. $\phi_{22} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2}$ (6.2.3)

AR(1) model. $\rho_k = \phi^k$ $\phi_{22} = \frac{\phi^2 - \phi^2}{1-\phi^2} = 0$

AR($p$) model: $\phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p}$

$$Cov\left(Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \ldots - \phi_p Y_{t-p}, Y_{t-k} - h(Y_{t-k+1}, Y_{t-k+2}, \cdots, Y_{t-1})\right)$$
$$= Cov(e_t, Y_{t-k} - h(Y_{t-k+1}, Y_{t-k+2}, \cdots, Y_{t-1})) = 0$$

Since $e_t$ is independent of $Y_{t-k}, Y_{t-k+1}, Y_{t-k+2}, \cdots, Y_{t-1}$. $\phi_{kk} = 0$ for $k > p$ (6.2.4)

MA(1) model $\phi_{22} = \frac{-\theta^2}{1+\theta^2+\theta^4}$ (6.2.5)

$$\phi_{kk} = -\frac{\theta^k(1-\theta^2)}{1-\theta^{2(k+1)}} \text{ for } k \geq 1 \quad (6.2.6)$$

$$\rho_j = \phi_{k1}\rho_{j-1} + \phi_{k2}\rho_{j-2} + \phi_{k3}\rho_{j-3} + \cdots + \phi_{kk}\rho_{j-k} \quad \text{for } j = 1, 2, \ldots, k \quad (6.2.7)$$

$$\left.\begin{array}{l} \phi_{k1} + \rho_1\phi_{k2} + \rho_2\phi_{k3} + \cdots + \rho_{k-1}\phi_{kk} = \rho_1 \\ \rho_1\phi_{k1} + \phi_{k2} + \rho_1\phi_{k3} + \cdots + \rho_{k-2}\phi_{kk} = \rho_2 \\ \vdots \\ \rho_{k-1}\phi_{k1} + \rho_{k-2}\phi_{k2} + \rho_{k-3}\phi_{k3} + \cdots + \phi_{kk} = \rho_k \end{array}\right\} \quad (6.2.8)$$

$\phi_{kk} = 0$ for $k > p$.

$$\phi_{kk} = \frac{\rho_k - \sum_{j=1}^{k-1}\phi_{k-1,j}\rho_{k-j}}{1-\sum_{j=1}^{k-1}\phi_{k-1,j}\rho_j} \quad (6.2.9) \qquad \phi_{k,j} = \phi_{k-1,j} - \phi_{kk}\phi_{k-1,k-j} \text{ for } j = 1,2,\ldots,k-1$$

$$\phi_{11} = \rho_1 \quad \phi_{22} = \frac{\rho_2 - \phi_{11}\rho_1}{1-\phi_{11}\rho_1} = \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} \quad \phi_{21} = \phi_{11} - \phi_{22}\phi_{11}\phi_{33} = \frac{\rho_3 - \phi_{21}\rho_2 - \phi_{22}\rho_1}{1-\phi_{21}\rho_2 - \phi_{22}\rho_1}$$

$\pm 2/\sqrt{n}$ critical limits on $\hat{\phi}_{kk}$

$$W_{t,k,j} = Y_t - \tilde{\phi}_1 Y_{t-1} - \cdots - \tilde{\phi}_k Y_{t-k} \quad (6.2.10)$$

**The Dickey-Fuller Unit-Root Test**

Under the null hypothesis that $\alpha = 1$, $X_t = Y_t - Y_{t-1}$.

$$\begin{aligned} Y_t - Y_{t-1} &= (\alpha-1)Y_{t-1} + X_t \\ &= aY_{t-1} + \phi_1 X_{t-1} + \cdots + \phi_k X_{t-k} + e_t \quad (6.4.1) \\ &= aY_{t-1} + \phi_1(Y_{t-1} - Y_{t-2}) + \cdots + \phi_k(Y_{t-k} - Y_{t-k-1}) + e_t \end{aligned}$$

AIC = – 2log(maximum likelihood) + $2k$ (6.5.1)

where $\begin{cases} k = p + q + 1 \text{ if the model contains an intercept or constant term and} \\ k = p + q \quad \text{otherwise.} \end{cases}$

The Kullback-Leibler divergence of $q_\theta$ from $p$ is defined by the formula

$$D(p, q_\theta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} p(y_1, y_2, \ldots, y_n)\log\left[\frac{p(y_1, y_2, \ldots, y_n)}{q_\theta(y_1, y_2, \ldots, y_n)}\right]dy_1 dy_2 \ldots dy_n$$

AIC estimates $E[D(p,q_{\hat{\theta}})]$, where $\hat{\theta}$ is the maximum likelihood estimator of the vector parameter $\theta$.

$\text{AIC}_c = \text{AIC} + \frac{2(k+1)(k+2)}{n-k-2}$ (6.5.2)

BIC = – 2log(maximum likelihood) + $k \log(n)$ (6.5.3)

ARMA(12,12) subset model useful for modeling some monthly seasonal time series: $Y_t = 0.8Y_{t-12} + e_t + 0.7e_{t-12}$ (6.5.4)

**The IMA(1,1) Model** $Y_t = Y_{t-1} + e_t - \theta e_{t-1}$ (5.2.5)

$Y_t = e_t + (1-\theta)e_{t-1} + (1-\theta)e_{t-2} + \cdots + (1-\theta)e_{-m} - \theta e_{-m-1}$ (5.2.6) $Var(Y_t) = [1+\theta^2+(1-\theta)^2(t+m)]\sigma_e^2$ (5.2.7)

$$Corr(Y_t, Y_{t-k}) = \frac{1-\theta+\theta^2+(1-\theta)^2(t+m-k)}{[Var(Y_t)Var(Y_{t-k})]^{1/2}}$$
$$\approx \sqrt{\frac{t+m-k}{t+m}} \qquad (5.2.8)$$
$$\approx 1 \qquad \text{for large } m \text{ and moderate } k$$

**The IMA(2,2) Model** $Y_t = 2Y_{t-1} - Y_{t-2} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$ (5.2.9) $\nabla^2 Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$

$$Y_t = e_t + \sum_{j=1}^{t+m} \psi_j e_{t-j} - [(t+m+1)\theta_1 + (t+m)\theta_2]e_{-m-1} - (t+m+1)\theta_2 e_{-m-2} \qquad (5.2.10)$$

where $\psi_j = 1 + \theta_2 + (1-\theta_1-\theta_2)j$ for $j = 1, 2, 3, \ldots, t+m$.

**The ARI(1,1) Model** $Y_t - Y_{t-1} = \phi(Y_{t-1} - Y_{t-2}) + e_t$ (5.2.11) or $Y_t = (1+\phi)Y_{t-1} - \phi Y_{t-2} + e_t$ (5.2.12)

where $|\phi| < 1$.†

$(1-\phi x)(1-x)(1+\psi_1 x+\psi_2 x^2+\psi_3 x^3+\cdots) = 1$ or $[1-(1+\phi)x+\phi x^2](1+\psi_1 x+\psi_2 x^2+\psi_3 x^3+\cdots) = 1$

$$-(1+\phi)+\psi_1 = 0$$
$$\phi-(1+\phi)\psi_1+\psi_2 = 0$$

and in general $\psi_k = (1+\phi)\psi_{k-1} - \phi\psi_{k-2}$ for $k \geq 2$ (5.2.14) with $\psi_0 = 1$ and $\psi_1 = 1+\phi$.

$$\psi_k = \frac{1-\phi^{k+1}}{1-\phi} \quad \text{for } k \geq 1 \qquad (5.2.15)$$

**General ARIMA(*p*,*d*,*q*)**

$$(1-\phi_1 x-\phi_2 x^2-\cdots-\phi^p x^p)(1-x)^d(1+\psi_1 x+\psi_2 x^2+\psi_3 x^3+\cdots) = (1-\theta_1 x-\theta_2 x^2-\theta_3 x^3-\cdots-\theta_q x^q) \qquad (5.2.13)$$

**Constant Terms in ARIMA Models**

$$W_t - \mu = \phi_1(W_{t-1}-\mu)+\phi_2(W_{t-2}-\mu)+\cdots+\phi_p(W_{t-p}-\mu) + e_t-\theta_1 e_{t-1}-\theta_2 e_{t-2}-\cdots-\theta_q e_{t-q}$$

or

$$W_t = \theta_0+\phi_1 W_{t-1}+\phi_2 W_{t-2}+\cdots+\phi_p W_{t-p} + e_t-\theta_1 e_{t-1}-\theta_2 e_{t-2}-\cdots-\theta_q e_{t-q}$$

$$\mu = \frac{\theta_0}{1-\phi_1-\phi_2-\cdots-\phi_p} \qquad (5.3.16)$$ or $$\theta_0 = \mu(1-\phi_1-\phi_2-\cdots-\phi_p) \qquad (5.3.17)$$

IMA(1,1) case with a constant term.

$Y_t = Y_{t-1}+\theta_0+e_t-\theta e_{t-1}$ or $W_t = \theta_0+e_t-\theta e_{t-1}$

$$Y_t = e_t+(1-\theta)e_{t-1}+(1-\theta)e_{t-2}+\cdots+(1-\theta)e_{-m}-\theta e_{-m-1} + (t+m+1)\theta_0 \qquad (5.3.18)$$

with a linear deterministic time trend with slope $\theta_0$.

Equivalently, the model can be expressed as $Y_t = Y_t' + \beta_0 + \beta_1 t$ $Y_t'$ is an IMA(1,1) series with $E(\nabla Y_t') = 0$ and $E(\nabla Y_t) = \beta_1$.

ARIMA(*p*,*d*,*q*) with constant terms: $Y_t = Y_t' + \mu_t$ where $\mu_t$ is deterministic polynomial time trend of order *d* and $Y_t'$ is ARIMA(*p*,*d*,*q*) with E[$Y_t'$] = 0.

## CHAP 7 PARAMETER ESTIMATION

### 7.1 The Method of Moments

| Model | Parameter Estimates | Estimating Equations |
|---|---|---|
| AR(1) | $\hat{\phi} = r_1$ (7.1.1) | |
| AR(2) | $\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2}$ and $\hat{\phi}_2 = \frac{r_2-r_1^2}{1-r_1^2}$ (7.1.2) | $r_1 = \phi_1 + r_1\phi_2$ and $r_2 = r_1\phi_1 + \phi_2$ |
| AR(*p*) | solve for $\hat{\phi}_j$ simultaneously using (7.1.3) | $\phi_1 + r_1\phi_2 + r_2\phi_3 + \cdots + r_{p-1}\phi_p = r_1$<br>$r_1\phi_1 + \phi_2 + r_1\phi_3 + \cdots + r_{p-2}\phi_p = r_2$ |

| | | |
|---|---|---|
| | | $\vdots$<br>$r_{p-1}\phi_1 + r_{p-2}\phi_2 + r_{p-3}\phi_3 + \cdots + \phi_p = r_p$ (7.1.3) |
| MA(1) | $\hat{\theta} = \frac{-1+\sqrt{1-4r_1^2}}{2r_1}$ (7.1.4) | $r_1 = \frac{\theta}{1+\theta^2}$ |
| ARMA(1,1) | $\hat{\phi} = \frac{r_2}{r_1}$ (7.1.5)<br>solve for $\hat{\theta}$ using (7.1.6) by retaining only the invertible solution. | $r_1 = \frac{(1-\theta\hat{\phi})(\hat{\phi}-\theta)}{1-2\theta\hat{\phi}+\theta^2}$ (7.1.6) |

| Model | Estimates of the Noise Variance $\sigma_e^2$ | Estimating Equations |
|---|---|---|
| AR(1) | $\hat{\sigma}_e^2 = (1-r_1^2)s^2$ (7.1.8) | Step 1. Estimate the process variance $\gamma_0$=Var($Y_t$) by<br>$s^2 = \frac{1}{n-1}\sum_{t=1}^{n}(Y_t - \bar{Y})^2$<br>Step 2. Equate relationship of parameters in chap 4 to $s^2$ to solve for parameters. |
| AR($p$) | $\hat{\sigma}_e^2 = (1-\hat{\phi}_1 r_1 - \hat{\phi}_2 r_2 - \cdots - \hat{\phi}_p r_p)s^2$ (7.1.7) | Same as above |
| MA($q$) | $\hat{\sigma}_e^2 = \frac{s^2}{1+\hat{\theta}_1^2+\hat{\theta}_2^2+\cdots+\hat{\theta}_q^2}$ (7.1.9) | Same as above |
| ARMA(1,1) | $\hat{\sigma}_e^2 = \frac{1-\hat{\phi}^2}{1-2\hat{\phi}\hat{\theta}+\hat{\theta}^2}s^2$ (7.1.10) | Same as above |

**7.2 Least Squares Estimation**

| Model | Parameter Estimates | Estimating Equations |
|---|---|---|
| AR(1) | $\mu = \frac{1}{(n-1)(1-\phi)}\left[\sum_{t=2}^{n} Y_t - \phi \sum_{t=2}^{n} Y_{t-1}\right]$ (7.2.3)<br>$\hat{\mu} \approx \frac{1}{1-\phi}(\bar{Y} - \phi\bar{Y}) = \bar{Y}$ (7.2.4)<br>$\hat{\phi} = \frac{\sum_{t=2}^{n}(Y_t-\bar{Y})(Y_{t-1}-\bar{Y})}{\sum_{t=2}^{n}(Y_{t-1}-\bar{Y})^2}$ | Step 1. Minimize conditional Sum-of-Squares<br>$S_c(\phi,\mu) = \sum_{t=2}^{n}[(Y_t-\mu)-\phi(Y_{t-1}-\mu)]^2$<br>Step 2. Solve for parameters. |
| AR(2) | $\hat{\mu} = \bar{Y}$ (7.2.5)<br>Other parameters by Yule-Walker equations | $S_c(\phi_1,\phi_2,\bar{Y}) = \sum_{t=3}^{n}[(Y_t-\bar{Y})-\phi_1(Y_{t-1}-\bar{Y})-\phi_2(Y_{t-2}-\bar{Y})]^2$ (7.2.6)<br>Yule-Walker equations<br>$r_1 = \phi_1 + r_1\phi_2$ (7.2.9)<br>$r_2 = r_1\phi_1 + \phi_2$ (7.2.10) |
| AR($p$) | $\hat{\mu} = \bar{Y}$ (7.2.5)<br>Other parameters by Yule-Walker equations | Similar to above |
| MA(1) | $e_1 = Y_1$<br>$e_2 = Y_2 + \theta e_1$<br>$e_3 = Y_3 + \theta e_2$<br>$\vdots$<br>$e_n = Y_n + \theta e_{n-1}$ (7.2.14) | $S_c(\theta) = \sum(e_t)^2 = \sum[Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \theta^3 Y_{t-3} + \cdots]^2$<br>(7.2.12) |
| MA($q$) | | Similar to above. Recursively obtain $e_t = e_t(\theta_1, \theta_2, \ldots, \theta_q)$<br>From $e_t = Y_t + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q}$ (7.2.15) |
| ARMA(1,1) | | Similar to above. Recursively obtain $e_t = e_t(\phi,\theta)$<br>From $e_t = Y_t - \phi Y_{t-1} + \theta e_{t-1}$ (7.2.17)<br>Minimize $S_c(\phi,\theta) = \sum_{t=2}^{n} e_t^2$ to avoid start-up problem |

| | | |
|---|---|---|
| ARMA($p$,$q$) | Numerical | Similar to above. $e_t = Y_t - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \cdots - \phi_p Y_{t-p} + \theta_1 e_{t-1} + \theta_2 e_{t-2} + \cdots + \theta_q e_{t-q}$ (7.2.18) with $e_p = e_{p-1} = \cdots = e_{p+1-q} = 0$ |

**7.3 Maximum Likelihood and Unconditional Least Squares**

| Model | Parameter Estimates | Estimating Equations |
|---|---|---|
| AR(1) | $Var(\hat{\phi}) \approx \frac{1-\phi^2}{n}$ (7.4.9) | $(2\pi\sigma_e^2)^{-1/2}\exp\left(-\frac{e_t^2}{\sigma_e^2}\right)$ for $-\infty < e_t < \infty$ <br> $(2\pi\sigma_e^2)^{-(n-1)/2}\exp\left(-\frac{1}{\sigma_e^2}\sum_{t=2}^{n} e_t^2\right)$ (7.3.1) <br> $Y_2 - \mu = \phi(Y_1 - \mu) + e_2$, $Y_3 - \mu = \phi(Y_2 - \mu) + e_3$, $\vdots$, $Y_n - \mu = \phi(Y_{n-1} - \mu) + e_n$ (7.3.2) <br> $L(\phi, \mu, \sigma_e^2) = (2\pi\sigma_e^2)^{-n/2}(1-\phi^2)^{1/2}\exp\left[-\frac{1}{2\sigma_e^2}S(\phi, \mu)\right]$ (7.3.4) <br> Unconditional Sum-of-squares <br> $S(\phi, \mu) = \sum_{t=2}^{n}[(Y_t - \mu) - \phi(Y_{t-1} - \mu)]^2 + (1-\phi^2)(Y_1 - \mu)$ (7.3.5) <br> $S(\phi, \mu) = S_c(\phi, \mu) + (1-\phi^2)(Y_1 - \mu)^2$ (7.3.8) <br> $\ell(\phi, \mu, \sigma_e^2) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma_e^2) + \frac{1}{2}\log(1-\phi^2) - \frac{1}{2\sigma_e^2}S(\phi, \mu)$ (7.3.6) |
| AR(2) | $Var(\hat{\phi}_1) \approx Var(\hat{\phi}_2) \approx \frac{1-\phi_2^2}{n}$ <br> $Corr(\hat{\phi}_1, \hat{\phi}_2) \approx -\frac{\phi_1}{1-\phi_2} = -\rho_1$ <br> (7.4.10) | Similar to above |
| MA(1) | $Var(\hat{\theta}) \approx \frac{1-\theta^2}{n}$ (7.4.11) | Similar to above |
| MA(2) | $Var(\hat{\theta}_1) \approx Var(\hat{\theta}_2) \approx \frac{1-\theta_2^2}{n}$ <br> $Corr(\hat{\theta}_1, \hat{\theta}_2) \approx -\frac{\theta_1}{1-\theta_2}$ (7.4.12) | Similar to above |
| ARMA(1,1) | $Var(\hat{\phi}) \approx \left[\frac{1-\phi^2}{n}\right]\left[\frac{1-\phi\theta}{\phi-\theta}\right]^2$ <br> $Var(\hat{\theta}) \approx \left[\frac{1-\theta^2}{n}\right]\left[\frac{1-\phi\theta}{\phi-\theta}\right]^2$ <br> $Corr(\hat{\phi}, \hat{\theta}) \approx \frac{\sqrt{(1-\phi^2)(1-\theta^2)}}{1-\phi\theta}$ (7.4.13) | Similar to above |

**7.4 Properties of the Estimates**

| Model | Variance or Standard Errors of Estimates | Remarks |
|---|---|---|
| AR(1) | $\sqrt{\hat{Var}(\hat{\phi})} \approx \sqrt{\frac{1-\hat{\phi}^2}{n}}$ | For large Samples, Standard error of Estimates formula same for Method of Moments (MoM), Least Squares, and MLE |
| AR(2) | $\sqrt{\hat{Var}(\hat{\phi}_1)} \approx \sqrt{\hat{Var}(\hat{\phi}_2)} \approx \sqrt{\frac{1-\hat{\phi}_2^2}{n}}$ | Same as above |
| AR($p$) | $\hat{\sigma}_e^2 = \left(1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2 - \cdots - \hat{\phi}_p r_p\right)s^2$ (7.1.7) | Same as above |
| MA(1) | MoM: $Var(\hat{\theta}) \approx \frac{1+\theta^2+4\theta^4+\theta^6+\theta^8}{n(1-\theta^2)^2}$ (7.4.14) <br> MLE and LSE: $\sqrt{\hat{Var}(\hat{\theta})} \approx \sqrt{\frac{1-\hat{\theta}^2}{n}}$ | Method of Moments (MoM) is different from Least Squares and MLE |

**7.5 Illustrations of Parameter Estimation**

**7.6 Bootstrapping ARIMA Models**

## CHAP 8 MODEL DIAGNOSTICS

### 8.1 Residual Analysis

| Model | Residual = actual − predicted | Residual autocorrelation |
|---|---|---|
| AR(2) with constant mean | $\hat{e}_t = Y_t - \hat{\phi}_1 Y_{t-1} - \hat{\phi}_2 Y_{t-2} - \hat{\theta}_0$ (8.1.2)<br>$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \theta_0 + e_t$ (8.1.1) | |
| ARMA($p$,$q$) | $\hat{e}_t = Y_t - \hat{\pi}_1 Y_{t-1} - \hat{\pi}_2 Y_{t-2} - \hat{\pi}_3 Y_{t-3} - \cdots$ (8.1.3)<br>$Y_t = \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \pi_3 Y_{t-3} + \cdots + e_t$ invertible form | |
| AR(1) | | $Var(\hat{r}_1) \approx \frac{\phi^2}{n}$ (8.1.5)<br>$Var(\hat{r}_k) \approx \frac{1-(1-\phi^2)\phi^{2k-2}}{n}$ for $k>1$ (8.1.6)<br>$Corr(\hat{r}_1, \hat{r}_k) \approx -sign(\phi)\frac{(1-\phi^2)\phi^{k-2}}{1-(1-\phi^2)\phi^{2k-2}}$ for $k>1$ (8.1.7)<br>$sign(\phi) = \begin{cases} 1 & \text{if } \phi>0 \\ 0 & \text{if } \phi=0 \\ -1 & \text{if } \phi<0 \end{cases}$ |
| AR(2) | | $Var(\hat{r}_1) \approx \frac{\phi_2^2}{n}$ (8.1.8)<br>$Var(\hat{r}_2) \approx \frac{\phi_2^2 + \phi_1^2(1+\phi_2)^2}{n}$ (8.1.9)<br>$Var(\hat{r}_k) \approx \frac{1}{n}$ for $k \geq 3$ (8.1.10) |

**Ljung_box Test:** $Q = n(\hat{r}_1^2 + \hat{r}_2^2 + \cdots + \hat{r}_K^2)$ (8.1.11)

**Modified Box-Pierce, or Ljung-Box, statistic is given by** $Q^* = n(n+2)\left(\frac{\hat{r}_1^2}{n-1} + \frac{\hat{r}_2^2}{n-1} + \cdots + \frac{\hat{r}_K^2}{n-K}\right)$ (8.1.12)

For large $n$, $Q$ has an approximate chi-square distribution with $K - p - q$ degrees of freedom.

### 8.2 Overfitting and Parameter Redundancy

**Overfitting Strategy**

The original (simpler) model will be confirmed if

1. the estimate of the additional parameter is not statistically different from zero, and
2. the estimates of parameters in common between considered models do not change significantly from the original estimates.

The implications for fitting and overfitting models are as follows:

1. Specify the original model **carefully**. If a simple model seems at all promising, check it out before trying a more complicated model.
2. When overfitting, **do not increase** the orders of both the AR and MA parts of the model simultaneously.

## CHAP 9 FORECASTING

### 9.1 Minimum Mean Square Error Forecasting

$\hat{Y}_t(\ell) = E(Y_{t+\ell}|Y_1, Y_2, \ldots, Y_t)$ (9.1.1)

### 9.2 Deterministic Trends

| **Model**<br>$Y_t = \mu_t + X_t$, (9.2.1)<br>*where* $E(X_t) = 0$ | **Forecast($\ell \geq 1$)**<br>$E(Y_{t+\ell}\|Y_1, Y_2, \ldots, Y_t) =$<br>$\hat{Y}_t(\ell) = \mu_{t+\ell}$ (9.2.2) | **Forecast Error**<br>$e_t(\ell) = Y_{t+\ell} - \hat{Y}_t(\ell)$ | **Forecast ErrorVariance**<br>$Var(e_t(\ell))$ |
|---|---|---|---|
| $\mu_t = \beta_0 + \beta_1 t$ | $\hat{Y}_t(\ell) = \beta_0 + \beta_1(t+\ell)$<br>(9.2.3) | $e_t(\ell) = X_{t+\ell}$<br>$E(e_t(\ell)) = E(X_{t+\ell}) = 0$ | $Var(X_{t+\ell}) = \gamma_0$ (9.2.4) |

| | | | |
|---|---|---|---|
| | | Forecasts are unbiased. | |
| Seasonal model $\mu_t=\mu_{t+12}$<br>$\mu_t=\beta_j$ $j=\mathrm{mod}(t,12)$<br>$j = 12$ if $\mathrm{mod}(t,12)=0$ | $\hat{Y}_t(\ell) = \mu_{t+12+\ell} = \hat{Y}_t(\ell+12)$ | $e_t(\ell) = X_{t+\ell}$<br>$E(e_t(\ell)) = E(X_{t+\ell}) = 0$<br>Forecasts are unbiased. | $Var(X_{t+\ell}) = \gamma_0$ (9.2.4) |
| Cosine Trend model<br>$\mu_t = \beta\cos(2\pi ft+\Phi)$(3.3.4)<br>$\mu_t = \beta_1\cos(2\pi ft)+\beta_2\sin(2\pi ft)$<br>(3.3.5)<br>where $\beta = \sqrt{\beta_1^2+\beta_2^2}$<br>$\Phi = \mathrm{atan}(-\beta_2/\beta_1)$ (3.3.6)<br>conversely $\beta_1 = \beta\cos(\Phi)$<br>$\beta_2 = -\beta\sin(\Phi)$ (3.3.7) | $\hat{\mu}_t$ | $e_t(\ell) = X_{t+\ell}$<br>$E(e_t(\ell)) = E(X_{t+\ell}) = 0$<br>Forecasts are unbiased. | $Var(X_{t+\ell}) = \gamma_0$ (9.2.4) |

**9.3 ARIMA Forecasting**

| **Model** | **Forecast($\ell \geq 1$)**<br>$E(Y_{t+\ell}\|Y_1,Y_2,\dots,Y_t)$ | **Forecast Error**<br>$e_t(\ell) = Y_{t+\ell} - \hat{Y}_t(\ell)$ | **Forecast Error Variance**<br>$Var(e_t(\ell))$ |
|---|---|---|---|
| AR(1) with nonzero mean<br>$Y_t - \mu = \phi(Y_{t-1}-\mu) + e_t$<br>(9.3.1) | $\hat{Y}_t(1) = \mu + \phi[Y_t - \mu]$ (9.3.6)<br>$\hat{Y}_t(\ell) = \mu + \phi[\hat{Y}_t(\ell-1) - \mu]$<br>(9.3.7)<br>$\hat{Y}_t(\ell) = \mu + \phi^\ell[Y_t - \mu]$ (9.3.8)<br>Since $\|\phi\| < 1$ for large $\ell$,<br>$\hat{Y}_t(\ell) \approx \mu$ (9.3.9) | $e_t(1) = e_{t+1}$ (9.3.10)<br>$E(e_{t+1}\|Y_1,Y_2,\dots,Y_t) = E(e_{t+1}) = 0$ (9.3.5)<br>$e_t(\ell) = e_{t+\ell} + \phi e_{t+\ell-1} + \cdots + \phi^{\ell-1}e_{t+1}$<br>(9.3.13)<br>$e_t(\ell) = e_{t+\ell} + \psi_1 e_{t+\ell-1} + \psi_2 e_{t+\ell-2} + \cdots + \psi_{\ell-1}e_{t+1}$<br>(9.3.14)<br>$E(e_t(\ell)) = 0$ Forecasts are unbiased. | $Var(e_t(1)) = \sigma_e^2$ (9.3.11)<br>$Var(e_t(\ell)) = \left[\frac{1-\phi^{2\ell}}{1-\phi^2}\right]\sigma_e^2$ (9.3.16)<br>$Var(e_t(\ell)) \approx \frac{\sigma_e^2}{1-\phi^2}$ for large $\ell$<br>(9.3.17)<br>For large $\ell$,<br>$Var(e_t(\ell)) \approx Var(Y_t) = \gamma_0$<br>(9.3.18) |
| ARMA($p$,$q$)<br>$Y_{t+\ell} = C_t(\ell) + I_t(\ell)$ for $\ell > 1$<br>(9.3.35)<br>truncated linear process<br>$I_t(\ell) = e_{t+\ell} + \psi_1 e_{t+\ell-1} + \psi_2 e_{t+\ell-2} + \cdots + \psi_{\ell-1}e_{t+1}$ for $\ell \geq 1$ (9.3.36) | $\hat{Y}_t(\ell) = E(C_t(\ell)\|Y_1,Y_2,\dots,Y_t) + E(I_t(\ell)\|Y_1,Y_2,\dots,Y_t) = C_t(\ell)$<br>$\hat{Y}_t(\ell) = \phi_1\hat{Y}_t(\ell-1) + \phi_2\hat{Y}_t(\ell-2) + \cdots + \phi_p\hat{Y}_t(\ell-p) + \theta_0$<br>$-\theta_1 E(e_{t+\ell-1}\|Y_1,Y_2,\dots,Y_t)$<br>$-\theta_2 E(e_{t+\ell-2}\|Y_1,Y_2,\dots,Y_t)$<br>$-\cdots-\theta_q E(e_{t+\ell-q}\|Y_1,Y_2,\dots,Y_t)$<br>(9.3.28)<br>$E(e_{t+j}\|Y_1,Y_2,\dots,Y_t) = \begin{cases} 0 & \text{for } j>0 \\ e_{t+j} & \text{for } j \leq 0 \end{cases}$<br>(9.3.29) | $e_t(\ell) = e_{t+\ell} + \psi_1 e_{t+\ell-1} + \psi_2 e_{t+\ell-2} + \cdots + \psi_{\ell-1}e_{t+1}$<br>(9.3.14)<br>$E[e_t(\ell)] = 0$ for $\ell \geq 1$ (9.3.37)<br>$e_t(\ell) = I_t(\ell)$<br>$= e_{t+\ell} + \psi_1 e_{t+\ell-1} + \psi_2 e_{t+\ell-2} + \cdots + \psi_{\ell-1}e_{t+1}$ | $Var(e_t(\ell)) = \sigma_e^2(1+\psi_1^2+\psi_2^2+\cdots+\psi_{\ell-1}^2)$<br>(9.3.15)<br>$Var(e_t(\ell)) = \sigma_e^2\sum_{j=0}^{\ell-1}\psi_j^2$<br>for $\ell \geq 1$ (9.3.38)<br>For large $\ell$,<br>$Var(e_t(\ell)) \approx \sigma_e^2\sum_{j=0}^{\infty}\psi_j^2 \approx \gamma_0$<br>(9.3.39) |
| ARMA(1,1) | $\hat{Y}_t(1) = \phi Y_t + \theta_0 - \theta e_t$ (9.3.30)<br>$\hat{Y}_t(2) = \phi\hat{Y}_t(1) + \theta_0$<br>$\hat{Y}_t(\ell) = \phi\hat{Y}_t(\ell-1) + \theta_0$ for $\ell \geq 2$<br>(9.3.31)<br>$\hat{Y}_t(\ell) = \mu + \phi^\ell(Y_t - \mu) - \phi^{\ell-1}e_t$ for $\ell \geq 1$<br>(9.3.32) | | |
| MA(1) | $\hat{Y}_t(1) = \mu - \theta E(e_t\|Y_1,Y_2,\dots,Y_t)$<br>(9.3.19)<br>$\hat{Y}_t(1) = \mu - \theta e_t$ (9.3.21)<br>$\hat{Y}_t(\ell) = \mu$ for $\ell > 1$<br>(9.3.22) | $e_t(1) = e_{t+1}$ | |
| Random Walk with Drift<br>$Y_t = Y_{t-1} + \theta_0 + e_t$ (9.3.23) | $\hat{Y}_t(1) = Y_t + \theta_0$ (9.3.24)<br>$\hat{Y}_t(\ell) = \hat{Y}_t(\ell-1) + \theta_0$ for $\ell \geq 1$<br>(9.3.25)<br>$\hat{Y}_t(\ell) = Y_t + \theta_0\ell$ for $\ell \geq 1$<br>(9.3.26) | $e_t(1) = Y_{t+1} - \hat{Y}_t(1) = e_{t+1}$<br>$e_t(\ell) = e_{t+1} + e_{t+2} + \cdots + e_{t+\ell}$ | $Var(e_t(\ell)) = \sigma_e^2\sum_{j=0}^{\ell-1}\psi_j^2 = \ell\sigma_e^2$<br>(9.3.27) |

| | | | |
|---|---|---|---|
| ARIMA($p$,1,$q$) or ARMA($p$+1,$q$) | $Y_t = \varphi_1 Y_{t-1} + \varphi_2 Y_{t-2} + \varphi_3 Y_{t-3} + \cdots + \varphi_p Y_{t-p} + \varphi_{p+1} Y_{t-p-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$ (9.3.40)<br>$\varphi_1 = 1 + \phi_1,\ \varphi_j = \phi_j - \phi_{j-1}$ for $j = 1, 2, \ldots, p$ and $\varphi_{p+1} = -\phi_p$ (9.3.41) | | |
| ARMA(1,1,1) | $\hat{Y}_t(1) = (1+\phi)Y_t - \phi Y_{t-1} + \theta_0 - \theta e_t$<br>$\hat{Y}_t(2) = (1+\phi)\hat{Y}_t(1) - \phi Y_t + \theta_0$<br>and<br>$\hat{Y}_t(\ell) = (1+\phi)\hat{Y}_t(\ell-1) - \phi\hat{Y}_t(\ell-2) + \theta_0$ (9.3.42) | | |
| ARIMA($p$,$d$,$q$) | | $e_t(\ell) = e_{t+\ell} + \psi_1 e_{t+\ell-1} + \psi_2 e_{t+\ell-2} + \cdots + \psi_{\ell-1} e_{t+1}$ for $\ell \geq 1$ (9.3.43)<br>$E(e_t(\ell)) = 0$ for $\ell \geq 1$ (9.3.44) | $Var(e_t(\ell)) = \sigma_e^2 \sum_{j=0}^{\ell-1} \psi_j^2$ for $\ell \geq 1$ (9.3.45) |
| IMA(1,1) | | $\psi_j = 1 - \theta$ for $j \geq 1$ | |
| IMA(2,2) | | $\psi_j = 1 + \theta_2 + (1 - \theta_1 - \theta_2)j$ for $j \geq 1$ | |
| ARI(1,1) | | $\psi_j = (1 - \phi^{j+1})/(1 - \phi)$ for $j \geq 1$ | |

## 9.4 Prediction Limits

| **Model** | **Forecast**($\ell \geq 1$)<br>$E(Y_{t+\ell} \mid Y_1, Y_2, \ldots, Y_t)$ | **Prediction Limits**<br>$\hat{Y}_t(\ell) \pm z_{1-\alpha/2}\sqrt{Var(e_t(\ell))}$ (9.4.2) |
|---|---|---|
| Deterministic $Y_t = \mu_t + X_t$, (9.2.1)<br>*where* $E(X_t) = 0$ $X_t$ is white noise | $\hat{Y}_t(\ell) = \mu_{t+\ell}$ | $\hat{Y}_t(\ell) \pm z_{1-\alpha/2}\sqrt{Var(e_t(\ell))}$ (9.4.2) |
| ARIMA($p$,$d$,$q$) | | $Var(e_t(\ell)) = \sigma_e^2 \sum_{j=0}^{\ell-1} \psi_j^2$ |
| AR(1) | | $Var(e_t(\ell)) = \left[\frac{1-\phi^{2\ell}}{1-\phi^2}\right]\sigma_e^2$ |

## 9.5 Forecasting Illustrations

## 9.6 Updating ARIMA Forecasts (Not on T162 Final!!!!)

$$Y_{t+\ell+1} = C_t(\ell+1) + e_{t+\ell+1} + \psi_1 e_{t+\ell} + \psi_2 e_{t+\ell-1} + \cdots + \psi_\ell e_{t+1}$$

$$\hat{Y}_{t+1}(\ell) = C_t(\ell+1) + \psi_\ell e_{t+1}$$

General updating equation: $\hat{Y}_{t+1}(\ell) = \hat{Y}_t(\ell+1) + \psi_\ell[Y_{t+1} - \hat{Y}_t(1)]$ (9.6.1)

## 9.7 Forecast Weights and Exponentially Weighted Moving Averages

$$\hat{Y}_t(1) = \pi_1 Y_t + \pi_2 Y_{t-1} + \pi_3 Y_{t-2} + \cdots \quad (9.7.1)$$

$$\pi_j = \begin{cases} \sum_{i=1}^{min(j,q)} \theta_i \pi_{j-i} + \varphi_j & \text{for } 1 \leq j \leq p+d \\ \sum_{i=1}^{min(j,q)} \theta_i \pi_{j-i} & \text{for } j > p+d \end{cases} \quad (9.7.2)$$

IMA(1,1) $\pi_1 = \theta\pi_0 + 1 = 1 - \theta$, $\pi_2 = \theta\pi_1 = \theta(1-\theta)$, and generally, $\pi_j = \theta\pi_{j-1}$ for $j > 1$.

$\pi_2 = \theta^{j-1}(1-\theta)$ for $j > 1$ (9.7.3)

$$\hat{Y}_t(1) = (1-\theta)Y_t + (1-\theta)\theta Y_{t-1} + (1-\theta)\theta^2 Y_{t-2} + \cdots \quad (9.7.4)$$

$\sum_{j=1}^{\infty} \pi_j = (1-\theta)\sum_{j=1}^{\infty} \theta^{j-1} = \frac{1-\theta}{1-\theta} = 1$ $\pi$-weights *decrease exponentially*.

So, $\hat{Y}_t(1)$ is **Exponentially Weighted Moving Average.**

$$\hat{Y}_t(1) = (1-\theta)Y_t + \theta\hat{Y}_{t-1}(1) \qquad (9.7.5)$$

$$\hat{Y}_t(1) = \hat{Y}_{t-1}(1) + (1-\theta)[Y_t - \hat{Y}_{t-1}(1)] \qquad (9.7.6)$$

### 9.8 Forecasting Transformed Series (Not on T162 Final!!!!)

| **Model** | **Forecast**($\ell \geq 1$)<br>$E(Y_{t+\ell} \mid Y_1, Y_2, \ldots, Y_t)$ |
|---|---|
| IMA(1,1) | $\hat{Y}_t(1) = Y_t - \theta e_t$ (9.8.1) $\quad \hat{W}_t(1) = -\theta e_t$ (9.8.3)<br>$\hat{Y}_t(\ell) = \hat{Y}_t(\ell-1)$ for $\ell > 1$ (9.8.2) $\quad \hat{W}_t(\ell) = 0$ for $\ell > 1$ (9.8.4)<br>$\hat{W}_t(1) = \hat{Y}_t(1) - Y_t$;<br>$\hat{W}_t(1) = -\theta e_t \leftrightarrow \hat{Y}_t(1) = Y_t - \theta e_t$ |
| Log-Transformed series | $E(Y_{t+\ell} \mid Y_t, Y_{t-1}, \ldots, Y_1) \geq \exp[E(Z_{t+\ell} \mid Z_t, Z_{t-1}, \ldots, Z_1)]$ (9.8.5)<br>$\exp\left\{\hat{Z}_t(\ell) + \frac{1}{2}Var[e_t(\ell)]\right\}$ (9.8.6)<br>If $Z_t$ ~Normal, then $Y_t = \exp(Z_t)$ ~LogNormal |

### 9.9 Summary of Forecasting with Certain ARIMA Models

| **Model** | **Forecast**($\ell \geq 1$)<br>$E(Y_{t+\ell} \mid Y_1, Y_2, \ldots, Y_t)$ | **Forecast Error** | **Forecast Error Variance** |
|---|---|---|---|
| AR(1) | $\hat{Y}_t(\ell) = \mu + \phi[\hat{Y}_t(\ell-1) - \mu]$ for $\ell \geq 1$<br>$= \mu + \phi^{\ell}(Y_t - \mu)$ for $l \geq 1$<br>$\hat{Y}_t(\ell) \approx \mu$ for large $\ell$ | $e_t(\ell) = e_{t+\ell} + \phi e_{t+\ell-1} + \cdots + \phi^{\ell-1}e_{t+1}$<br>$\psi_j = \phi^j$ for $j > 0$ | $Var(e_t(\ell)) = \sigma_e^2\left[\frac{1-\phi^{2\ell}}{1-\phi^2}\right]$<br>$Var(e_t(\ell)) \approx \frac{\sigma_e^2}{1-\phi^2} = \gamma_0$ for large $\ell$ |
| MA(1) | $\hat{Y}_t(1) = \mu - \theta e_t$<br>$\hat{Y}_t(\ell) = \mu$ for $l > 1$ | $e_t(1) = e_{t+1}$<br>$e_t(\ell) = e_{t+\ell} - \theta e_{t+\ell-1}$ for $\ell > 1$<br>$\psi_j = \begin{cases} -\theta & \text{for } j = 1 \\ 0 & \text{for } j > 1 \end{cases}$ | $Var(e_t(\ell)) = \begin{cases} \sigma_e^2 & \text{for } \ell = 1 \\ \sigma_e^2(1+\theta^2) & \text{for } \ell > 1 \end{cases}$ |
| IMA(1,1) with constant term $\theta_0$ | $\hat{Y}_t(\ell) = \hat{Y}_t(\ell-1) + \theta_0 - \theta e_t$<br>$= Y_t + \ell\theta_0 - \theta e_t$<br>$\hat{Y}_t(1) = (1-\theta)Y_t + (1-\theta)\theta Y_{t-1}$<br>$+ (1-\theta)\theta^2 Y_{t-2} + \cdots$<br>(EWMA with $\theta_0 = 0$)<br>• If $\theta_0 \neq 0$, forecasts follow a straight line with slope $\theta_0$<br>• If $\theta_0 = 0$, $\hat{Y}_t(\ell) = Y_t - \theta e_t$ | $e_t(\ell) = e_{t+\ell} + (1-\theta)e_{t+\ell-1} +$<br>$(1-\theta)e_{t+\ell-2} +$<br>$\cdots + (1-\theta)e_{t+1}$ for $\ell \geq 1$ | $Var(e_t(\ell)) = \sigma_e^2[1 + (\ell-1)(1-\theta)^2]$<br>$\psi_j = 1 - \theta$ for $j > 0$ |
| IMA(2,2) with constant term $\theta_0$ | $\left.\begin{aligned} \hat{Y}_t(1) &= 2Y_t - Y_{t-1} + \theta_0 - \theta_1 e_t - \theta_2 e_{t-1} \\ \hat{Y}_t(2) &= 2\hat{Y}_t(1) - Y_t + \theta_0 - \theta_2 e_t \\ \hat{Y}_t(\ell) &= 2\hat{Y}_t(\ell-1) - \hat{Y}_t(\ell-2) + \theta_0 \text{ for } \ell > 2 \end{aligned}\right\}$<br>(9.9.1)<br>$\hat{Y}_t(\ell) = A + B\ell + \frac{\theta_0}{2}\ell^2$ (9.9.2)<br>where<br>$A = 2\hat{Y}_t(1) - \hat{Y}_t(2) + \theta_0$ (9.9.3)<br>$B = \hat{Y}_t(2) - \hat{Y}_t(1) - \frac{3}{2}\theta_0$ (9.9.4)<br>• If $\theta_0 \neq 0$, forecasts follow a quadratic curve in $\ell$<br>• If $\theta_0 = 0$, forecasts form a straight line with slope of $\hat{Y}_t(2) - \hat{Y}_t(1)$ | | $\psi_j = 1 + \theta_2 + (1 - \theta_1 - \theta_2)j$ for $j > 0$ (9.9.5) |

| | | | |
|---|---|---|---|
| | • Forecasting special case with $\theta_1 = 2\omega$ and $\theta_2 = -\omega^2$ is a Double Exponential Smoothing with smoothing constant $1-\omega$ | | |

## CHAP 10 Seasonal Models

### 10.1 Seasonal ARIMA Models

| **Model** | | **Char Polynomial** | **Auto-correlation** |
|---|---|---|---|
| MA($Q$)$_s$ | $Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2s} - \cdots - \Theta_Q e_{t-Qs}$ (10.1.1) | $\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \cdots - \Theta_Q x^{Qs}$ (10.1.2) | $\rho_{ks} = \frac{-\Theta_k + \Theta_1\Theta_{k+1} + \Theta_2\Theta_{k+2} + \cdots + \Theta_{Q-k}\Theta_Q}{1+\Theta_1^2+\Theta_2^2+\cdots+\Theta_Q^2}$ for $k = 1,2,\cdots,Q$(10.1.3) |
| AR(1)$_{12}$ | $Y_t = \Phi Y_{t-12} + e_t$ (10.1.4) | | $\rho_k = \Phi\rho_{k-12}$ for $k \geq 1$ (10.1.5) $\rho_{12k} = \Phi^k$ for $k = 1,2,\cdots$ (10.1.6) |
| AR($P$)$_s$ | $Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2s} + \cdots + \Phi_P Y_{t-Ps} + e_t$ (10.1.7) | $\Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \cdots - \Phi_P x^{Ps}$ (10.1.8) | |
| AR(1)$_s$ | | | $\rho_{ks} = \Phi^k$ $for$ $k = 1,2,\cdots$ (10.1.9) |

### 10.2 Multiplicative Seasonal ARMA Models

| **Model** | | **Char Polynomial** | **Auto-correlation** |
|---|---|---|---|
| MA(1)x(1)$_{12}$ | $Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$ (10.2.1) | $(1-\theta x)(1-\Theta x^{12})$ $1 - \theta x - \Theta x^{12} + \theta\Theta x^{13}$ | $\gamma_0 = (1-\theta^2)(1+\Theta^2)\sigma_e^2$ (10.2.2) $\rho_1 = -\frac{\theta}{1+\theta^2}$ (10.2.3) $\rho_{11} = \rho_{13} = \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)}$ (10.2.4) $\rho_{12} = -\frac{\Theta}{1+\Theta^2}$ (10.2.5) |
| ARMA($p$,$q$)X($P$,$Q$)$_s$ | | AR char poly $\phi(x)\Phi(x)$ (10.2.6) $\left.\begin{matrix}\phi(x) = 1 - \phi_1 x - \phi_2 x^2 - \cdots - \phi_p x^p \\ \Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \cdots - \Phi_P x^{Ps}\end{matrix}\right\}$ MA char poly $\theta(x)\Theta(x)$ (10.2.7) $\left.\begin{matrix}\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \cdots - \theta_q x^q \\ \Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \cdots - \Theta_Q x^{Qs}\end{matrix}\right\}$ | (10.2.2) (10.2.3) (10.2.4) |
| ARMA(0,1)X(1,0)$_{12}$ | $Y_t = \Phi Y_{t-12} + e_t - \theta e_{t-1}$ (10.2.8) | | $\gamma_1 = \Phi\gamma_{11} - \theta\sigma_e^2$ (10.2.9) $\gamma_k = \Phi\gamma_{k-12}$ for $k \geq 2$ (10.2.10) $\left.\begin{matrix}\gamma_0 = \left[\frac{1+\theta^2}{1-\Phi^2}\right]\sigma_e^2 \\ \rho_{12k} = \Phi^k \text{ for } k \geq 1 \\ \rho_{12k-1} = \rho_{12k+1} = \left(-\frac{\theta}{1+\theta^2}\Phi^k\right) \; for \; k = 0,1,2,\cdots\end{matrix}\right\}$ (10.2.11) |

### 10.3 Nonstationary Seasonal ARIMA Models

$\nabla_s Y_t = Y_t - Y_{t-s}$ (10.3.1) $Y_t = S_t + e_t$ (10.3.2) $S_t = S_{t-s} + \varepsilon_t$ (10.3.3) $\{e_t\}$ and $\{\varepsilon_t\}$ are mutually independent white noise series. If $\sigma_\varepsilon \ll \sigma_e, \{S_t\}$ would model a slowly changing se asonal component.

MA(1)$_s$ : $\nabla_s Y_t = S_t - S_{t-s} + e_t - e_{t-s} = \varepsilon_t + e_t - e_{t-s}$ (10.3.4)

$Y_t = M_t + S_t + e_t$ (10.3.5) $S_t = S_{t-s} + \varepsilon_t$ (10.3.6) $M_t = M_{t-1} + \xi_t$ (10.3.7)

$\{e_t\}$, $\{\varepsilon_t\}$, and $\{\xi_t\}$ are mutually independent white noise series.

ARMA(0,1)x(0,1)$_s$ :

$$\nabla\nabla_s Y_t = \nabla(M_t - M_{t-s} + \varepsilon_t + e_t - e_{t-s}) = (\xi_t + \varepsilon_t + e_t) - (\varepsilon_{t-1} + e_{t-1}) + (\xi_{t-s} + e_{t-s}) + e_{t-s-1} \qquad (10.3.8)$$

ARMA($p$,$q$)x($P$,$Q$)$_s$ : $W_t = \nabla^d \nabla_s^D Y_t$ (10.3.9)

### 10.4 Model Specification, Fitting, and Checking: $\nabla_{12}\nabla Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$ (10.4.10)

### 10.5 Forecasting Seasonal Models

| **Model** | | **Forecast** | **Forecast Error Variance** $Var(e_t(\ell))$ |
|---|---|---|---|
| ARMA(0,1,1)x(1,0,1)$_{12}$ | $Y_t - Y_{t-1} = \Phi(Y_{t-12} - Y_{t-13}) + e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$<br>(10.5.1)<br>$Y_t = Y_{t-1} - \Phi Y_{t-12} - \Phi Y_{t-13} + e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$<br>(10.5.2) | $\hat{Y}_t(1) = Y_t + \Phi Y_{t-11} - \Phi Y_{t-12} - \theta e_t - \Theta e_{t-11} + \theta\Theta e_{t-12}$<br>(10.5.3)<br>$\hat{Y}_t(2) = \hat{Y}_t(1) + \Phi Y_{t-10} - \Phi Y_{t-11} - \Theta e_{t-10} + \theta\Theta e_{t-11}$<br>(10.5.4)<br>$\hat{Y}_t(\ell) = \hat{Y}_t(\ell-1) + \Phi\hat{Y}_t(\ell-12) - \Phi\hat{Y}_t(\ell-13)$<br>for $\ell > 13$ (10.5.5) | |
| AR(1)$_{12}$ | $Y_t = \Phi Y_{t-12} + e_t$ (10.5.6) | $\hat{Y}_t(\ell) = \Phi\hat{Y}_t(\ell-12)$ (10.5.7)<br>$\hat{Y}_t(\ell) = \Phi^{k+1} Y_{t+r-11}$ (10.5.8)<br>$\ell = 12k + r + 1$ with $0 \le r < 12$ and $k = 0,1,2,\cdots$<br>$\psi_j = \begin{cases} \Phi^{j/12} & \text{for } j = 0,12,24,\cdots \\ 0 & otherwise \end{cases}$<br>(10.5.9) | $Var(e_t(\ell)) = \left[\frac{1-\Phi^{2k+2}}{1-\Phi^2}\right]\sigma_e^2$<br>(10.5.10)<br>$k$ integer part of $(\ell-1)/12$. |
| MA(1)$_{12}$ | $Y_t = e_t - \Theta e_{t-12} + \theta_0$ (10.5.11) | $\left.\begin{array}{l} \hat{Y}_t(1) = -\Theta e_{t-10} + \theta_0 \\ \hat{Y}_t(1) = -\Theta e_{t-10} + \theta_0 \\ \vdots \\ \hat{Y}_t(12) = -\Theta e_t + \theta_0 \end{array}\right\}$ (10.5.12)<br>$\hat{Y}_t(\ell) = \theta_0 \quad for\ \ell > 12$ (10.5.13) | $\psi_0 = 1, \psi_{12} = -\Theta$, and $\psi_j = 0$ otherwise<br>$Var(e_t(\ell)) = \begin{cases} \sigma_e^2 & 1 \le \ell \le 12 \\ (1+\Theta^2)\sigma_e^2 & 12 < \ell \end{cases}$<br>(10.5.14) |
| ARIMA(0,0,0)x(0,1,1)$_{12}$ | $Y_t - Y_{t-12} = e_t - \Theta e_{t-12}$ (10.5.15)<br>$Y_{t+\ell} = Y_{t+\ell-12} + e_{t+\ell} - \Theta e_{t+\ell-12}$<br>Inverted model version:<br>$Y_t = (1-\Theta)(Y_{t-12} + \Theta Y_{t-24} + \Theta^2 Y_{t-36} + \cdots) + e_t$ | $\left.\begin{array}{l} \hat{Y}_t(1) = Y_{t-11} - \Theta e_{t-11} \\ \hat{Y}_t(2) = Y_{t-10} - \Theta e_{t-10} \\ \vdots \\ \hat{Y}_t(12) = Y_t - \Theta e_t \end{array}\right\}$ (10.5.16)<br>$\hat{Y}_t(\ell) = \hat{Y}_t(\ell-12)$ for $\ell > 12$ (10.5.17)<br>$\left.\begin{array}{l} \hat{Y}_t(1) = (1-\Theta)\sum_{j=0}^{\infty}\Theta^j Y_{t-11-12j} \\ \hat{Y}_t(2) = (1-\Theta)\sum_{j=0}^{\infty}\Theta^j Y_{t-10-12j} \\ \vdots \\ \hat{Y}_t(12) = (1-\Theta)\sum_{j=0}^{\infty}\Theta^j Y_{t-12j} \end{array}\right\}$<br>(10.5.18) | $\psi_j = 1 - \Theta$ for $j = 12,24,\cdots,$<br>$Var(e_t(\ell)) = [1 + k(1-\Theta)^2]\sigma_e^2$<br>(10.5.19)<br>where $k$ is the integer part of $(\ell - 1)/12$. |
| ARIMA(0,1,1)x(0,1,1)$_{12}$ | $Y_t = Y_{t-1} + Y_{t-12} - Y_{t-13} + e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$ (10.5.20) | $\left.\begin{array}{l} \hat{Y}_t(1) = Y_t + Y_{t-11} - Y_{t-12} - \theta e_t - \Theta e_{t-11} + \theta\Theta e_{t-12} \\ \hat{Y}_t(2) = \hat{Y}_t(1) + Y_{t-10} - Y_{t-11} - \Theta e_{t-10} + \theta\Theta e_{t-11} \\ \vdots \\ \hat{Y}_t(12) = \hat{Y}_t(11) + Y_t - Y_{t-1} - \Theta e_t - \theta\Theta e_{t-1} \\ \hat{Y}_t(13) = \hat{Y}_t(12) + \hat{Y}_t(1) - Y_t + \theta\Theta e_t \end{array}\right\}$<br>(10.5.21)<br>$\hat{Y}_t(\ell) = \hat{Y}_t(\ell-1) + \hat{Y}_t(\ell-12) + \hat{Y}_t(\ell-13)$<br>for $\ell > 13$ (10.5.22)<br>Alternate forecast representation:<br>$\hat{Y}_t(\ell) = A_1 + A_2\ell + \sum_{j=0}^{6} B_{1j}\cos\left(\frac{2\pi j\ell}{12}\right) + B_{2j}\sin\left(\frac{2\pi j\ell}{12}\right)$ (10.5.23)<br>$A's$ and $B's$ are dependent on $Y_t, Y_{t-1}, \cdots$, or, determined from initial forecasts<br>$\hat{Y}_t(1), \hat{Y}_t(2), \cdots, \hat{Y}_t(13)$ | |

Prediction Limits: obtained precisely as in the nonseasonal case.