King Fahd University of Petroleum and Minerals

Department of Mathematics and Statistics

Math 601
Final Exam – 2016–2017 (162)
Wednesday, May 24, 2017

Allowed Time: 120 minutes

Instructor: **Dr. Boubaker Smii**

Name: ______________________________

ID #: ____________________

Section #: __________ Serial Number: __________

Instructions:

1. Write clearly and legibly. You may lose points for messy work.
2. **Show all your work**. No points for answers without justification !

Question #	Grade	Maximum Points
1		11
2		14
3		16
4		09
5		20
6		10
Total:		80

Exercise 1:

Let X be a random variable.

i)- Show that, if X has exponential distribution of rate $\lambda > 0$. then

$$P(X > t + s) = P(X > t)\, P(X > s), \qquad s, t \geq 0. \tag{a}$$

ii)- Show that the equality (a) is equivalent to:

$$P(X > t + s \mid X > s) = P(X > t), \qquad s, t \geq 0. \tag{b}$$

iii)- Assume now that $X(t) = N(t)$, $t > 0$ has the Poisson distribution with parameter $\lambda\, t$, $\lambda > 0$ i.e: $P(N(t) = n) = e^{-\lambda\, t}\, \frac{(\lambda\, t)^n}{n!}$. Find the expectation of $N(t)$.

Exercise 2:

Let B_t be a standard Brownian motion.

1- Write down the stochastic differential equation corresponding to the process $Y_t = B_t^4$ and deduce $\mathbb{E}(B_t^4)$.

2-Write down the stochastic differential equation corresponding to the process $Y_t = t\,B_t$.

3-Verify that $X_t = e^{B_t - \frac{t}{2}}$ satisfies the stochastic differential equation: $dX_t = X_t\,dB_t$.

Exercise 3:

Let X_t, Y_t be Itô processes in $\mathbb{R}$.

1)- Prove that:

$$X_t\,Y_t = X_0\,Y_0 + \int_0^t Y_s\,dX_s + \int_0^t X_s\,dY_s + \int_0^t dX_s\,dY_s$$

2)- Let $F_t = \exp(-\,\alpha\,B_t + \frac{1}{2}\,\alpha^2\,t),\ \alpha\, \in\, \mathbb{R}$.

i)- Find dF_t.

ii)- Given that: $dY_t = r\,dt + \alpha\,Y_t\,dB_t,\ r\, \in\, \mathbb{R}$. Prove that $Y_t = Y_0\,F_t^{-1} + r\,F_t^{-1}\displaystyle\int_0^t F_s\,ds$ (**Hint**: Use 1)).

Exercise 4: Use Itô's formula to write the following stochastic processes X_t on the standard form.
a)- $X_t = B_t^2$, where B_t is 1-dimensional Brownian motion.

b)- $X_t = 2 + t + e^{B_t}$, where B_t is 1-dimensional Brownian motion.

c)- $X_t = B_1^2(t) + B_2^2(t)$, where (B_1, B_2) is a 2-dimensional Brownian motion.

Exercise 5: To describe the motion of a pendulum with small, random perturbations in its environment we consider the stochastic differential equation :

$$U_t'' + \Big(1 + \epsilon W_t\Big) U_t = 0; \qquad U_0,\, U_0' \;\; given, \tag{c}$$

where W_t is a one-dimensional white noise, ϵ a positive constant.

1- Show that the stochastic differential equation (c) can be written in the following form:

$$dX_t = K\, X_t\, dt - \epsilon\, L\, X_t\, dB_t, \tag{d}$$

where X_t, K, L are suitable matrices and B_t a Brownian motion.

2- Show that Y_t solves a stochastic Volterra equation of the form

$$Y_t = Y_0 + \, Y_0'\; t \, + \int_0^t \; a(t,r)\, Y_r\, dr \, + \int_0^t \; \gamma(t,r)\, Y_r\, dB_r\,, \tag{e}$$

where $a(t,r) = r - t$ and $\gamma(t,r) = \epsilon\,(r-t)$.

<u>Exercise 6:</u> A- The Black-Scholes-Merton model for growth with uncertain rate of return, is the value of §1 after time t, invested in a saving account. It is described by the following stochastic differential equation:

$$dX_t = \mu\, X_t\, dt + \sigma\, X_t\, dB_t,\ \ \mu, \sigma > 0 \tag{f}$$

Prove that the solution of the SDE (f) is given by a Geometric Brownian motion.

B-Let S_t be the price of a stock at time t. Suppose that stock price is modelled as a geometric Brownian motion $S_t = S_0\, e^{\mu\, t + \sigma\, B_t}$, where B_t is a standard Brownian motion.

Suppose that the parameter values are $\mu = 0.055$ and $\sigma = 0.07$.
Given that $S_5 = 100$, find the probability that S_{10} is greater than 150. (you may express the result as $\Phi(\alpha)$, where Φ is the standard Normal distribution function and α a real number.)