King Fahd University of Petroleum & Minerals
Department of Math. & Stat.

Final Exam - Math 568 (162)   Time: 3 hours

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Name: ID #

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Please show all work. No credit for a result without work

Problem 1	/5
Problem 2	/5
Problem 3	/10
Problem 4	/10
Problem 5	/5

Problem # 1. (5 marks) Let $u(x, y, z)$ be the temperature at a point $X = (x, y, z)$ of the ball $B_2(0) = \left\{X \in \mathbb{R}^3 \ / \ x^2 + y^2 + z^2 < 4\right\}$. If

$$\begin{cases} \Delta u(x, y, z) = 0, & \text{in } B_2(0) \\ u(x, y, z) = x^2 + y^2 + z^2 + z, & \text{on } \partial B_2(0), \end{cases}$$

1) find the temperature at the center: $u(0)$
2) find the quantity of heat q at the level $\{x^2 + y^2 + z^2 = 1\}$, where $q = \int_{\partial B_1(0)} u(X) dS_X$

Problem # 2. (5 marks) Let Ω be a bounded domain of $\mathbb{R}^2$. Given the problem

$$\begin{cases} u_t(x,y,t) - k\Delta u(x,y,t) + u^2(x,y,t) = 0 & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times [0,T] \\ u(x,y,0) = -x^2 + xy - y^2 & \text{in } \Omega \end{cases}$$

where $k > 0$. Show that

$$u(x,y,t) \leq 0, \forall (x,y,t) \text{ in } \Omega \times (0,T).$$

Problem # 3. (10 marks) Find $u(x, y, t)$, if u satisfies

$$\begin{cases} u_t(x, y, t) - 4\Delta u(x, y, t) + 2u(x, y, t) = 0 & \text{in } \mathbb{R}^2 \times (0, +\infty) \\ u(x, y, 0) = 2 + xy & \text{in } \mathbb{R}^2 \end{cases}$$

Hint: you may consider $v = ue^{\alpha t}$, for an appropriate α.

Problem # 4. (10 marks) Solve the problem

$$\begin{cases} u_{tt}(x,y,z,t) - \Delta u(x,y,z,t) = 0 & \text{in } \mathbb{R}^3 \times (0,+\infty) \\ u(x,y,z,0) = 2 + xz, \;\; u_t(x,y,z,0) = y^2 & \text{in } \mathbb{R}^3 \end{cases}$$

Problem # 5. (5 marks) Use the energy method to show that the problem

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } \Omega \times (0, +\infty) \\ u = g & \text{on } \partial\Omega \times [0, +\infty) \\ u(x,0) = u_0(x), \;\; u_t(x,0) = u_1(x) & \text{in } \Omega \end{cases}$$

has at most one solution. Here Ω is a bounded and regular domain of $\mathbb{R}^n$.