King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics

Math 531 (Real Analysis) Final Exam Spring 2016(162)- 120 minutes

MRF **Solution**

Notation: $\mathbb{R}$ = the real numbers, $\mathbb{N}$ = the natural numbers, m = Lebesgue measure.
Instructions: Work any three complete problems or any six different parts

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(1) (a) For $x \in (0, +\infty)$, set $f_n(x) = \frac{e^{\sin(x^2/n)}}{1+x}$ for each $n \in \mathbb{N}$.

(i) Evaluate with proof $\lim\limits_{n\to\infty} \int\limits_0^n (f_n(x))^2 dx$.

(ii) Evaluate with proof $\lim\limits_{n\to\infty} \int\limits_0^n f_n(x) dx$.

Solution: (i) We first rewrite the integrals so that the limits of integration are independent of n as follow:

$$\lim_{n\to\infty} \int_0^n (f_n(x))^2 dx = \lim_{n\to\infty} \int_0^\infty \chi_{(0,n)}(f_n(x))^2 dx.$$

Now $\chi_{(0,n)}(f_n(x))^2 = \chi_{(0,n)} \frac{e^{2\sin(x^2/n)}}{(1+x)^2} \to \frac{1}{(1+x)^2}$ as $n \to \infty$, and for each $n \in \mathbb{N}$, $\left|\chi_{(0,n)}(f_n(x))^2\right| \le \frac{e^2}{(1+x)^2}$ for each $x \in (0, \infty)$. Further, $\int\limits_0^\infty \frac{e^2}{(1+x)^2} dx = e^2$. The Lebesgue Dominated Convergence Theorem then gives that

$$\lim_{n\to\infty} \int_0^\infty \chi_{(0,n)}(f_n(x))^2 dx = \int_0^\infty \lim_{n\to\infty} \chi_{(0,n)}(f_n(x))^2 dx = \int_0^\infty \frac{1}{(1+x)^2} = 1.$$

(ii) Again we rewrite the integrals so that the limits of integration are independent of n, but here we note that $\chi_{(0,n)} f_n(x) \to \frac{1}{1+x} \notin L^1(\mathbb{R}, m)$, and the convergence is not monotone. But if we use Fatou's lemma, we get,
$\liminf\limits_{n\to\infty} \int\limits_0^\infty \chi_{(0,n)} f_n(x) dx \ge \int\limits_0^\infty \liminf\limits_{n\to\infty} \chi_{(0,n)} f_n(x) dx = \int\limits_0^\infty \frac{1}{1+x} = \infty$. Hence we have $\lim\limits_{n\to\infty} \int\limits_0^\infty \chi_{(0,n)} f_n(x) dx = \infty$.

(b) Let f and g be nonnegative integrable functions on $[0, 1]$ with

$$\int_0^1 f(x) dx = \int_0^1 g(x) dx = 1.$$

Let $A = \{x \in [0,1] : f(x) \leq 3\}$ and $B = \{x \in [0,1] : g(x) \leq 3\}$. Show that $m(A \cap B) \geq \frac{1}{3}$.

Solution: We have $f \in L^1([0,1])$ with $||f||_1 = 1$. Let A' denote the complement of A in $[0,1]$, so $A' = \{x \in [0,1] : f(x) > 3\}$. By Chebyshev's inequality (and the nonnegativity of f), $m(A') \leq \frac{||f||_1}{3} = \frac{1}{3}$, and the same is, of course, true of g; $m(B') \leq \frac{1}{3}$. Thus $m(A' \cup B') \leq m(A') + m(B') \leq 2/3$. Now $m(A \cap B) = 1 - m((A \cap B)') = 1 - m(A' \cup B') \geq 1 - \frac{2}{3} = \frac{1}{3}$.

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(2) Identify which of the following statements is true and which is false. If a statement is true, give reason. If a statement is false, provide a counterexample

(a) (i) Let $(X, \mathcal{M}, \mu)$ be a measure space and f be a measurable function, then $f = g$ $a.e.$ implies g is measurable.

Solution: False. Let E be a subset of a set of measure zero that does not belong to $\mathcal{M}$. Let $f = 0$ on X and $g = \chi_E$. Then $f = g$ $a.e.$ on X while f is measurable and g is not.

(ii) Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L^1([0,1])$. Then $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable over $[0,1]$.

Solution: False. Consider the sequence $f_n = n\chi_{[0,1/n]}$. Clearly, $||f_n||_1 = 1$ for all $n \geq 1$ and therefore $\{f_n\}$ is bounded in $L^1([0,1])$. However,

$$\int_{[0,1/n]} f_n = 1,$$

for all n and therefore cannot be uniformly integrable over $[0,1]$

(b) If $f, g \in L^2(\mathbb{R})$, then $\lim\limits_{t\to\infty} \int f(x)g(tx)dm = 0$.

Solution: True. Since $f, g \in L^2(\mathbb{R})$, then there exist $0 < M < \infty$ and $0 < N < \infty$ such that $\left(\int |f(x)|^2\right)^{\frac{1}{2}} < M$ and $\left(\int |g(x)|^2\right)^{\frac{1}{2}} < N$. Now, by Hölder inequality we have $|\int f(x)g(tx)dm| \leq \left(\int |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int |g(tx)|^2 dx\right)^{\frac{1}{2}} = \left(\int |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int |g(y)|^2 \frac{1}{t} dy\right)^{\frac{1}{2}} = \frac{1}{t^{1/2}} \left(\int |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int |g(y)|^2 dy\right)^{\frac{1}{2}} < \frac{MN}{t^{1/2}}$. Hence $\lim\limits_{t\to\infty} \int f(x)g(tx)dm = 0$.

3(a) Let $\{E_n\}$ be a sequence of measurable subsets of $[0,1]$ and suppose that $m(E_n) \leq 1/n$. Show that if $f = \sum \chi_{E_n}/n$ and if $1 \leq p < \infty$ then $f \in L^p([0,1])$.

Solution: $||f||_p = \left(\int_{[0,1]} |f|^p\right)^{1/p} = \left(\int_{[0,1]} |\sum \chi_{E_n}/n|^p\right)^{1/p} \leq^{Minkowski}$
$\sum \left(\int_{[0,1]} |\chi_{E_n}/n|^p\right)^{1/p} = \sum \left(\int_{E_n} |1/n|^p\right)^{1/p} = \sum \left(1/n^p m(E_n)\right)^{1/p} \leq^{m(E_n)\leq 1/n}$

$\sum \left(1/n^{p+1}\right)^{1/p} = \sum \left(1/n^{1+1/p}\right)$. Since $1 + 1/p > 1$, then by p-serious test $||f||_p \leq \sum \left(1/n^{1+1/p}\right) < \infty$.

(b) (i) Let $f \in BV([a,b])$. Show that if $f \geq c$ on $[a,b]$ for some constant $c > 0$, then $\frac{1}{f} \in BV([a,b])$.

Solution: Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Then

$$V(\frac{1}{f}, \mathcal{P}) = \sum_{k=1}^{n} \Big| \frac{1}{f(x_k)} - \frac{1}{f(x_{k-1})} \Big| = \sum_{k=1}^{n} \frac{|f(x_k) - f(x_{k-1})|}{|f(x_k)f(x_{k-1})|}.$$

Since $f \geq c > 0$,

$$\frac{|f(x_k) - f(x_{k-1})|}{|f(x_k)f(x_{k-1})|} \leq \frac{|f(x_k) - f(x_{k-1})|}{c^2}.$$

It follows that

$$V(\frac{1}{f}, \mathcal{P}) \leq \frac{1}{c^2} \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| = \frac{1}{c^2} V(f, \mathcal{P}) \leq \frac{1}{c^2} TV(f).$$

Since $TV(f) < \infty$, $TV(\frac{1}{f}) < \infty$.

(ii) Let f be a real-valued function on $[a,b]$ satisfying the Lipschitz condition on $[a,b]$. Show that f is absolutely continuous on $[a,b]$.

Solution: The Lipschitz condition on $[a,b]$:

$$\exists \ K > 0: \ \ \forall x, y \in [a,b], \ \ |f(x) - f(y)| \leq K|x - y|.$$

Given any $\epsilon > 0$. Let $\delta = \frac{\epsilon}{K}$. Let $\{[c_i, d_i] : i, 1, \cdots, n\}$ be a family of disjoint subintervals of $[a,b]$ with $\sum_{i=1}^{n}(d_i - c_i) < \delta$. Then, by the Lipschitz condition, we have

$$\sum_{i=1}^{n} |f(c_k) - f(d_k)| \leq \sum_{i=1}^{n} K(d_k - c_k) \leq K \sum_{i=1}^{n}(d_k - c_k) < K.\frac{\epsilon}{K} = \epsilon.$$

Thus f is absolutely continuous on $[a,b]$.

4(a) Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$. If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

Solution: Suppose μ and ν are positive measures. If $\mu \perp \nu$, then there are disjoint measurable sets A and B such that $X = A \cup B$ and $\mu(B) = 0 = \nu(A)$.

For any measurable set E, we have $(\mu\wedge\nu)(E) = (\mu\wedge\nu)(E\cap A)+(\mu\wedge\nu)(E\cap B) = min(\mu(E\cap A), \nu(E\cap A)) + min(\mu(E\cap B), \nu(E\cap B)) = 0$. Conversely, suppose $\mu \wedge \nu = 0$. If $\mu(E) = \nu(E) = 0$ for all measurable sets, then $\mu = \nu = 0$ and $\mu \perp \nu$. Thus we may assume that $\mu(E) = 0 < \nu(E)$ for some E. If $\nu(E^c) = 0$, it follows that $\nu \perp \nu$. On the other hand, if $\nu(E^c) > 0$, then $\mu(E^c) = 0$ so $\mu(X) = \mu(E) + \mu(E^c) = 0$. Thus $\mu = 0$ and we still have $\mu \perp \nu$.

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of real-valued measurable functions on X such that, for each natural number n, $\mu\Big(\{x \in X : |f_n(x) - f_{n+1}(x)| > 1/2^n\}\Big) < 1/2^n$. Show that $\{f_n\}$ is pointwise convergent *a.e.* on X. (Hint: Use the Borel-Cantelli Lemma.)

Solution: Let $E_n = \{x \in X : |f_n(x) - f_{n+1}(x)| > 1/2^n\}$. Then $\{E_n\}_{n=1}^{\infty}$ is a countable collection of measurable sets with $\sum_{n=1}^{\infty} \mu(E_n) < \sum_{n=1}^{\infty} 1/2^n = 1$. Hence, by the Borel-Cantelli Lemma, almost all $x \in X$ belong to at most a finite number of the $E_n's$. Thus for each natural number n and for almost all $x \in X$, we have $|f_n(x) - f_{n+1}(x)| \leq 1/2^n$. So $\{f_n\}$ is is pointwise convergent *a.e.* on X.

Dr. M. R. Alfuraidan