King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 531 (Real Analysis) Major Exam I Spring 2016(162)- 120 minutes

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Notation: $\mathbb{R}$ = the real numbers, $\mathbb{N}$ = the natural numbers, m = Lebesgue measure.
Instructions: Work any five problems

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(1) (a) What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is measurable?

Solution: An extended real-valued function f defined on $\mathbb{R}$ is said to be Lebesgue measurable, or simply measurable, provided it satisfies one of the following four equivalent conditions:

(i) For each real number c, the set $\{x : f(x) > c\}$ is measurable.

(ii) For each real number c, the set $\{x : f(x) \geq c\}$ is measurable.

(iii) For each real number c, the set $\{x : f(x) < c\}$ is measurable.

(iv) For each real number c, the set $\{x : f(x) \leq c\}$ is measurable.

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is increasing (i.e. $f(x) \leq f(y)$ whenever $x \leq y$) then it is measurable.

Solution: (1) If f is increasing, the set $\{x \in \mathbb{R} : f(x) > a\}$ is an interval for all a, hence measurable. Therefore, by the definition (see (a) above), the function f is measurable.
(2) Let D be the set of discontinuities of f. Then D is countable, hence of measure zero. The restriction $f|_D$ is measurable on D because every subset of D is measurable, and the restriction $f|_{R\sim D}$ is measurable on $R \sim D$ because it is continuous. Therefore, f is measurable (see Proposition 5 - Section 3,1).

(c) Suppose that $f : [0, 1] \to \mathbb{R}$ is measurable and that there is $\delta > 0$ such that, for each $n \in \mathbb{N}$, $m\{x : |f(x)| \leq 1/n\} \geq \delta$.

(i) Explain why $\{x : |f(x)| \leq 1/n\}$ is measurable.

(ii) Explain why there is at least one $s \in [0, 1]$ such that $f(s) = 0$.

Solution:(i) Since f is measurable, $|f|$ is measurable as its the composition of continuous function $g(x) = |x|$ with a measurable function f) and $\{x : |f(x)| \leq 1/n\} = |f|^{-1}[0, 1/n]$. Since $[0, 1/n]$ is a Borel set, $|f|^{-1}[0, 1/n]$ is measurable.

(ii) Let $E_n = \{x : |f(x)| \leq 1/n\}$. Then $E_n \supset E_{n+1}$, $\cap E_n = \{x : f(x) = 0\}$,

$E_1 \subset [0,1]$. So $m\{x : f(x) = 0\} = \lim_{n\to\infty} m(E_n) \geq \delta$ (Excision property of m). Then $\{x : f(x) = 0\} \neq \emptyset$ (Since $m(\emptyset) = 0$). So $\exists\, s$ so that $f(s) = 0$.

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(2) (a) For a measurable subset $E \subseteq \mathbb{R}$, and simple function $\varphi : \mathbb{R} \to \mathbb{R}$, how is the (Lebesgue) integral $\int_E \varphi dm$ defined?

Solution: For a simple function φ defined on a set of finite measure E, we define the integral of φ over E by

$$\int_E \varphi = \sum_{i=1}^{n} a_i m(E_i),$$

where $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ and $E_i = \{x \in E : \varphi(x) = a_i\}$.

(b) State Fatou's Lemma for a sequence of measurable functions.

Solution: Let $\{f_n\}$ be a sequence of measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\int_E f = \int_E \lim f_n \leq \liminf \int_E f_n.$$

(c) State the Monotone Convergence Theorem.

Solution: Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If $\{f_n\} \to f$ pointwise a.e. on E, then

$$\lim_{n\to\infty} (\int_E f_n) = \int_E (\lim_{n\to\infty} f_n) = \int_E f.$$

(d) Prove that Fatou's Lemma implies the Monotone Convergence Theorem.

Solution: According to Fatou's Lemma,

$$\int_E f \leq \liminf(\int_E f_n).$$

Also, notice that if f is a nonnegative measurable function on E and E_0 is a subset of E of measure zero, then

$$\int_E f = \int_{E \sim E_0} f \quad (*).$$

However, for each n, $f_n \leq f$ a.e. on E (note that f is measurable), and by the monotonicity of integration for nonnegative measurable functions and $(*)$, $\int_E f_n \leq \int_E f$. Therefore,

$$\limsup \int_E f_n \leq \int_E f.$$

Hence $\int_E f = \lim_{n\to\infty} \int_E f_n$.

(3) Identify which of the following statements is true and which is false. If a statement is true, give reason. If a statement is false, provide a counterexample

(a) If f is a bounded real-valued function on $[0,1]$ which is Lebesgue integrable then f is Riemann integrable.

Solution: False. Consider the Dirichlet function $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0,1]; \\ 0, & x \in [0,1] \sim \mathbb{Q}. \end{cases}$

(b) Suppose that (E_n) is a sequence of pairwise disjoint measurable subsets of $[0,1]$. Then $\lim_{n\to\infty} m(E_n) = 0$.

Solution: True, indeed since $E_n \subset [0,1]$ $\forall n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} E_n \subset [0,1]$. By monotonicity of the measure $m(\bigcup_{n=1}^{\infty} E_n) \leq m([0,1]) = 1$. Hence $\sum_{n=1}^{\infty} m(E_n) \leq 1$ since m is countably additive. Thus $\lim_{n\to\infty} m(E_n) = 0$.

(c) If $f(x) = \int_{\mathbb{R}} \frac{(\sin t)^2}{t^2+x^2} dt$, then $\lim_{x\to\infty} f(x) = 0$.

Solution: $\frac{(\sin t)^2}{t^2+x^2} \leq \frac{1}{t^2+x^2}$ for all t. Hence by monotonicity of Riemann integrable functions $\int_{\mathbb{R}} \frac{(\sin t)^2}{t^2+x^2} dt \leq \int_{\mathbb{R}} \frac{1}{t^2+x^2} dt = \frac{1}{x^2} \int_{\mathbb{R}} \frac{1}{(\frac{t}{x})^2+1} dt = \frac{1}{x^2}[\tan^{-1} \frac{t}{x}]_{-\infty}^{+\infty} = \frac{\pi}{x^2} < \infty$. Since $f(x)$ is positive and limit of $\frac{\pi}{x^2}$ goes to zero as x goes to ∞, then $\lim_{x\to\infty} f(x) = 0$.

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(4) (a) State Egoroff's Theorem.

Assume E has finite measure. Let $\{f_n\}$, be a sequence of measurable functions on E that converges pointwise on E to the real-valued function f. Then for each $\epsilon > 0$, there is a closed set F contained in E such that $\{f_n\} \to f$ uniformly on F and $m(E \sim F) < \epsilon$.

(b) Let f be a real-valued measurable function defined on $[0,1]$. Prove that for each $\epsilon > 0$ there is a measurable set $E_\epsilon \subseteq [0,1]$ so that $m([0,1] \sim E_\epsilon) < \epsilon$ and so that f is bounded on E_ϵ.

Solution (1) (Using Egoroff's Theorem) Let $f_n = f\chi_{(|f| \leq n)}$. Since f is measurable, $\{x : |f(x)| \leq n\} = f^{-1}[-n,n]$ is measurable. So, $\chi_{(|f|\leq n)}$ is a measurable function and hence $f\chi_{(|f|\leq n)}$, the product of two measurable functions is measurable. If $|f(x)| < N$ then $f_n(x) = f(x)$ for all $n \geq N$. So, $\lim_{n\to\infty} f_n(x) = f(x)$ $\forall x$. Then by Egoroff's Theorem $\forall\ \epsilon > 0\ \exists\ E_\epsilon \subset [0,1]$ such that $m([0,1] \sim E_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on E_ϵ. Since $f_n \to f$ uniformly, in particular, $\exists\ N$ such that $|f_n(x) - f(x)| < 1$ for all $n \geq N$ and $x \in E_\epsilon$. Thus $|f(x)| < 1 + |f_n(x)| \leq N + 1$ on E_ϵ.

Or (2) Let $E_n = \{x : |f(x)| \geq n\}$. Then $E_{n+1} \subset E_n$, $\cap E_n = \emptyset$ and $m(E_1) \leq m[0,1] = 1$. Since f is real valued function, then $\forall\ \epsilon > 0\ \exists\ N$ such that $m(E_N) < \epsilon$. And $|f(x)| \leq N$ on $[0,1] \sim E_N$.

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(5) Suppose that f is integrable on $[0,1]$. Let $p_n(x) = x^n$, $n \in \mathbb{N}$.

(a) State why, for each n, fp_n is measurable and integrable on $[0,1]$.

Solution: p_n is continuous on $[0,1]$ and so p_n is measurable. Then fp_n, the product of two measurable functions is measurable. Moreover, $|p_n| \leq 1$ on $[0,1]$, so $|fp_n| \leq |f|$ and since f is integrable so is each fp_n.

(b) Prove that $\lim\limits_{n\to\infty} \int_{[0,1]} f.p_n dm = 0$.

Solution: Now $\lim\limits_{n\to\infty} f(x)p_n(x) = 0$ unless $x = 1$ or $|f(x)| = \infty$. So, $\lim\limits_{n\to\infty} f(x)p_n(x) = 0$ a.e. Since $|fp_n| \leq |f|$ we may apply the Dominated Convergence Theorem to get $\lim\limits_{n\to\infty} \int_{[0,1]} fp_n dm = \int_{[0,1]} \lim\limits_{n\to\infty} fp_n dm = 0$.

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(6) (a) State the Dominated Convergence Theorem.

Solution Let $\{f_n\}$ be a sequence of measurable functions on E. Suppose there is a function g that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \leq g$ on E for all n. If $\{f_n\} \to f$ pointwise a.e. on E, then f is integrable over E and $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$.

(b) Use the Dominated Convergence Theorem to find

$$\lim_{n\to\infty} \int_0^\infty f_n dm,$$

where for each $n \geq 1$ the function $f_n : [0,\infty) \to \mathbb{R}$ is defined by

$$f_n(x) = \frac{x \sin \pi n x}{1 + nx^3}.$$

Solution: For $n \geq 1$, we have $|f_n(x)| = |\frac{x \sin \pi nx}{1+nx^3}| \leq \frac{x}{1+nx^3} \leq \frac{x}{nx^3} = \frac{1}{nx^2} \leq \frac{1}{x^2}$ for all $x \in (0,\infty)$. Also note that the function $\frac{1}{x^2}$ is integrable over $[0,\infty)$ ($\int_{[0,\infty)} \frac{1}{x^2} = \int_{(0,\infty)} \frac{1}{x^2} < \infty$).

Thus, by the Dominated Convergence Theorem and the squeezing Theorem, we have

$$\lim_{n\to\infty} \int_{(0,\infty)} f_n = \int_{(0,\infty)} 0 = 0.$$

Notice that $\int_E f = \int_{E\sim E_0} f$ if $m(E_0) = 0$.

(7) (a) State Beppo Levis Theorem.

Solution Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on E. If the sequence of integrals $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converges pointwise on E to a measurable function f that is finite a.e on E and

$$\lim_{n\to\infty} \int_E f_n = \int_E f < \infty.$$

(b) Use Beppo Levis Theorem, and the fact that $\sum\limits_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$, to prove that

$$\int\limits_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}.$$

Solution: First notice that $\frac{x}{e^x-1} = \frac{xe^{-x}}{1-e^{-x}}$. Now, using the Geometric serious $1 + a + a^2 + ... + a^n = \frac{1-a^{n+1}}{1-a}$, if $|a| < 1$, we have $\frac{1}{1-e^{-x}} = \sum\limits_{n=0}^{\infty} e^{-nx}$ for $x > 0$. So,

$$\frac{x}{e^x - 1} = \frac{xe^{-x}}{1 - e^{-x}} = \sum_{n=0}^{\infty} xe^{-(n+1)x}.$$

Let $f(x) = \frac{x}{e^x-1}$ and define the sequence (f_n) by $f_n(x) = f\chi_{(0,n]}$ for each $n \geq 1$. Notice that (f_n) is an increasing sequence of nonnegative measurable functions (f_n is the product of two measurable functions ($\chi_{(0,n]}$ is measurable since $(0, n]$ is measurable and f is measurable since it is continuous on $(0, \infty)$). Moreover, $f_n \to f$ a.e. on $[0, \infty)$. Using Beppo Levis Theorem and integration by parts, we have $\lim\limits_{n\to\infty} \int\limits_0^\infty f_n dx = \int_0^\infty f dx = \int\limits_0^\infty \frac{x}{e^x-1} dx = \int\limits_0^\infty \sum\limits_{n=0}^\infty xe^{-(n+1)x} dx = \sum\limits_{n=0}^\infty \int\limits_0^\infty xe^{-(n+1)x} dx = \sum\limits_{n=0}^\infty \frac{1}{(n+1)^2} = \sum\limits_{n=1}^\infty \frac{1}{(n)^2} = \frac{\pi^2}{6}$.

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