King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 531 (Real Analysis) Final Exam Spring 2016(162)- 120 minutes

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Notation: $\mathbb{R}$ = the real numbers, $\mathbb{N}$ = the natural numbers, m = Lebesgue measure.
Instructions: Work any three complete problems or any six different parts

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(1) (a) For $x \in (0, +\infty)$, set $f_n(x) = \frac{e^{\sin(x^2/n)}}{1+x}$ for each $n \in \mathbb{N}$.

(i) Evaluate with proof $\lim_{n\to\infty} \int_0^n (f_n(x))^2 dx$.

(ii) Evaluate with proof $\lim_{n\to\infty} \int_0^n f_n(x) dx$.

(b) Let f and g be nonnegative integrable functions on $[0, 1]$ with

$$\int_0^1 f(x)dx = \int_0^1 g(x)dx = 1.$$

Let $A = \{x \in [0, 1] : f(x) \leq 3\}$ and $B = \{x \in [0, 1] : g(x) \leq 3\}$. Show that $m(A \cap B) \geq \frac{1}{3}$.

(2) Identify which of the following statements is true and which is false. If a statement is true, give reason. If a statement is false, provide a counterexample

(a) (i) Let $(X, \mathcal{M}, \mu)$ be a measure space and f be a measurable function, then $f = g$ $a.e.$ implies g is measurable.

(ii) Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is bounded in $L^1([0, 1])$. Then $\{f_n\}_{n\in\mathbb{N}}$ is uniformly integrable over $[0, 1]$.

(b) If $f, g \in L^2(\mathbb{R})$, then $\lim_{t\to\infty} \int f(x)g(tx)dm = 0$.

3(a) Let $\{E_n\}$ be a sequence of measurable subsets of $[0, 1]$ and suppose that $m(E_n) \leq 1/n$. Show that if $f = \sum \chi_{E_n}/n$ and if $1 \leq p < \infty$ then $f \in L^p([0, 1])$.

(b) (i) Let $f \in BV([a, b])$. Show that if $f \geq c$ on $[a, b]$ for some constant $c > 0$, then $\frac{1}{f} \in BV([a, b])$.

(ii) Let f be a real-valued function on $[a, b]$ satisfying the Lipschitz condition on $[a, b]$. Show that f is absolutely continuous on $[a, b]$.

4(a) Let μ and ν be finite signed measures. Define $\mu \wedge \nu = \frac{1}{2}(\mu + \nu - |\mu - \nu|)$. If μ and ν are positive measures, show that they are mutually singular if and only if $\mu \wedge \nu = 0$.

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$ be a sequence of real-valued measurable functions on X such that, for each natural number n, $\mu\Big(\{x \in X : |f_n(x) - f_{n+1}(x)| > 1/2^n\}\Big) < 1/2^n$. Show that $\{f_n\}$ is pointwise convergent $a.e.$ on X. (Hint: Use the Borel-Cantelli Lemma.)

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