King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics

Math 531 (Real Analysis) Major Exam II Spring 2016(162)- 120 minutes

ID#:________________ NAME:________________________________

Notation: $\mathbb{R}$ = the real numbers, $\mathbb{N}$ = the natural numbers, m = Lebesgue measure.
Instructions: Work any three problems.

---

(1) (a) Let f be extended real-valued measurable function on a measurable set D. If $\langle E_n \rangle_{n\in\mathbb{N}}$ is an increasing sequence of measurable sets such that $\bigcup_{n\in\mathbb{N}} E_n = D$, then $\int_D f dm = \lim_{n\to\infty} \int_{E_n} f dm$.

(b) Let $\mathcal{F}$ be the family of functions f on $[0,1]$, each of which is continuous over $[0,1]$ and has $|f| \leq 1$. Is $\mathcal{F}$ uniformly integrable over $[0,1]$?

(2) (a) Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of measurable functions on $[0,1]$ such that $\langle f_n \rangle_{n\in\mathbb{N}}$ converges in measure to a function f on $[0,1]$. Prove that

$$\lim_{n\to\infty} \int_{[0,1]} \sin(f_n(x))dm(x) = \int_{[0,1]} \sin(f(x))dm(x).$$

(b) Suppose that f and g are integrable on $[0,1]$ and that $\lim_{h\to 0^+} \frac{1}{h} \int_x^{x+h} f(t)dt = \lim_{h\to 0^+} \frac{1}{h} \int_x^{x+h} g(t)dt$ for a.e. x. Prove that $f = g$ a.e.

(3) (a) Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, and let $E \subset [a,b]$ with $m(E) = 0$. Prove that $m(f(E)) = 0$.

(b) Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of functions of bounded variation on $[0,1]$ and suppose that $\exists\ C$ such that $V(f_n, [0,1]) \leq C\ \forall n$. Prove that if $\lim_{n\to\infty} f_n(x) = f(x)\ \forall\ 0 \leq x \leq 1$, then f is of bounded variation on $[0,1]$.

(4) (a) Let $f(x) = \begin{cases} 0 & x = 0 \\ 1/x & 0 < x \leq 1. \end{cases}$

Find a sequence of functions of bounded variation which converges pointwise to f, but show that $f \notin BV[0,1]$.

(b) Suppose that for each n, $f_n \in AC[0,1]$, f_n is monotone increasing on $[0,1]$ and $f_n(0) = 0$. If $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges for each $0 \leq x \leq 1$. Show that $f \in AC[0,1]$.

Dr. M. R. Alfuraidan