King Fahd University of Petroleum and Minerals
Department of Mathematics & Statistics
Math 531 (Real Analysis) Major Exam I Spring 2016(162)- 120 minutes

ID#:________________ NAME:________________________________

Notation: $\mathbb{R}$ = the real numbers, $\mathbb{N}$ = the natural numbers, m = Lebesgue measure.
Instructions: Work any five problems

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(1) (a) What does it mean to say that a function $f : \mathbb{R} \to \mathbb{R}$ is measurable? (2pts)

(b) Prove that if $f : \mathbb{R} \to \mathbb{R}$ is increasing (i.e. $f(x) \le f(y)$ whenever $x \le y$) then it is measurable. (3pts)

(c) Suppose that $f : [0,1] \to \mathbb{R}$ is measurable and that there is $\delta > 0$ such that, for each $n \in \mathbb{N}$, $m\{x : |f(x)| \le 1/n\} \ge \delta$. (5pts)

(i) Explain why $\{x : |f(x)| \le 1/n\}$ is measurable.

(ii) Explain why there is at least one $s \in [0,1]$ such that $f(s) = 0$.

(2) (a) For a measurable subset $E \subseteq \mathbb{R}$, and simple function $\varphi : \mathbb{R} \to \mathbb{R}$, how is the (Lebesgue) integral $\int_E \varphi dm$ defined? (2pts)

(b) State Fatou's Lemma for a sequence of measurable functions. (2pts)

(c) State the Monotone Convergence Theorem. (2pts)

(d) Prove that Fatou's Lemma implies the Monotone Convergence Theorem. (4pts)

(3) Identify which of the following statements is true and which is false. If a statement is true, give reason. If a statement is false, provide a counterexample

(a) If f is a bounded real-valued function on $[0,1]$ which is Lebesgue integrable then f is Riemann integrable. (2pts)

(b) Suppose that (E_n) is a sequence of pairwise disjoint measurable subsets of $[0,1]$. Then $\lim_{n\to\infty} m(E_n) = 0$. (4pts)

(c) If $f(x) = \int_{\mathbb{R}} \frac{(\sin t)^2}{t^2+x^2} dt$, then $\lim_{x\to\infty} f(x) = 0$. (4pts)

(4) (a) State Egoroff's Theorem. (3pts)

(b) Let f be a real-valued measurable function defined on $[0,1]$. Prove that for each $\epsilon > 0$ there is a measurable set $E_\epsilon \subseteq [0,1]$ so that $m([0,1] \sim E_\epsilon) < \epsilon$ and so that f is bounded on E_ϵ. (7pts)

(5) Suppose that f is integrable on $[0,1]$. Let $p_n(x) = x^n$, $n \in \mathbb{N}$.

(a) State why, for each n, fp_n is measurable and integrable on $[0,1]$. (5pts)

(b) Prove that $\lim\limits_{n\to\infty} \int_{[0,1]} f.p_n dm = 0$. (5pts)

(6) (a) State the Dominated Convergence Theorem. (3pts)

(b) Use the Dominated Convergence Theorem to find (7pts)

$$\lim_{n\to\infty} \int_0^\infty f_n dm,$$

where for each $n \geq 1$ the function $f_n : [0,\infty) \to \mathbb{R}$ is defined by

$$f_n(x) = \frac{x \sin \pi n x}{1 + n x^3}.$$

(7) (a) State Beppo Levis Theorem. (3pts)

(b) Use Beppo Levis Theorem, and the fact that $\sum\limits_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$, to prove that (7pts)

$$\int_0^\infty \frac{x}{e^x - 1} dx = \frac{\pi^2}{6}.$$

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