**King Fahd University of Petroleum and Minerals**
**Department of Mathematics and Statistics**
**MATH411 - Advanced Calculus II**
**Final Exam – Semester 162**

**Exercise 1**

Let

$$f(x,y) = \begin{cases} \dfrac{xyz}{(x^2+y^2+z^2)^\alpha} & \text{if } (x,y,z) \neq (0,0,0) \\ 0 & \text{if } (x,y,y) = (0,0,0) \end{cases}$$

where $\alpha \in \mathbb{R}$ is a constant. Show that $f$ is differentiable at $(0,0,0)$ if and only if $\alpha < 1$.

**Exercise 2**

Let $u = xyf(\frac{x+y}{xy})$, where $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Show that $u$ satisfies the equation

$$x^2\frac{\partial u}{\partial x} - y^2\frac{\partial u}{\partial y} = g(x,y)u$$

and find $g(x,y)$.

**Exercise 3**

For $x \in \mathbb{R}^n \setminus \{0\}$, let $f(x) = g(||x||)$ where $g$ is $\mathcal{C}^2$ on $(0, \infty)$. Show that

$$\frac{\partial^2 f}{\partial^2 x_1} + \ldots + \frac{\partial^2 f}{\partial^2 x_n} = g''(r) + \frac{n-1}{r} g'(r)$$

where $r = ||x||$.

Deduce that $f(x) = ||x||^{2-n}$, $n \geq 3$, is harmonic.

**Exercise 4**

Let $f(x,y) = e^{xy}\sin(x+y)$. Find the Taylor polynomial of $f$ of order 3 about the point $(0,0)$.

**Exercise 5**

Find the critical points of $f(x, y, z) = x^3 - y^3 + z^2 - 3x + 9y$ and determine their nature.

**Exercise 6**

Find the maximum of $x^2y^2z^2$ on the sphere $x^2 + y^2 + z^2 = R^2$.

**Exercise 7**

Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}$ be a bounded function. Define the following

1. $f$ is Riemann integrable
2. $f$ is Darboux integrable.

Give an example of a function $f$ on $[0,1] \times [0,1]$ which is not Riemann integrable.

**Exercise 8**

Let $f : \Omega \to \mathbb{R}$ be an integrable function over $\Omega$ a bounded simple subset of $\mathbb{R}^n$

Show that

1. If $f$ is almost everywhere zero, then $\int_\Omega f = 0$.
2. If $f \geq 0$ and $\int_\Omega f = 0$, then $f$ is almost everywhere zero.