King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
MATH411 - Advanced Calculus II
Exam I – Semester 162

Exercise 1

Let A and B be two compact subsets of $\mathbb{R}^n$. Show that $A \times B$ is a compact subset of $\mathbb{R}^{2n}$.

Exercise 2

Let $f : \Omega \to \mathbb{R}^m$ be uniformly continuous on $\Omega \subset \mathbb{R}^n$. Show that

(a) If (x_k) is a Cauchy sequence, then $(f(x_k))$ is a Cauchy sequence

(b) If Ω is bounded, then $f(\Omega)$ is bounded.

Exercise 3

Show that a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous if and only if, for each V open subset of $\mathbb{R}^m$, $f^{-1}(V)$ is open in $\mathbb{R}^n$.

Exercise 4

Determine whether the limit exists, and if it does find it

(a) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{x - y^4}{x^3 - y^4}$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} xy \log(x^2 + y^2)$

Exercise 5

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix. Define $q : \mathbb{R}^n \to \mathbb{R}$ by

$$q(x) =< Ax, x >$$

Show that q is differentiable and find $dq(x)$. (Hint: find $q(x + th)$)

Exercise 6

Let S be s nonempty subset of $\mathbb{R}^n$. The distance from the point $x \in \mathbb{R}^n$ to the set S is defined by

$$d(x,S) = \inf\{||x-y|| : y \in S\}$$

Show that

(a) $d(x,S) = 0$ if and only if $x \in \overline{S}$.

(b) $d(x,S) = d(x,\overline{S})$

(c) $|d(x,S) - d(y,S)| \leq ||x-y||$ for every $x, y \in \mathbb{R}^n$; consequently $d(.,S)$ is uniformly continuous on $\mathbb{R}^n$.

(d) If S is compact and $c \geq 0$, then $\{x \in \mathbb{R}^n : d(x,S) \leq c\}$ is compact.

(e) If S is compact, then there exists $y_0 \in S$ such that $d(x,S) = ||x - y_0||$.