KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 345 : FINAL EXAM, T 162, MAY 22, 2017

Time : $08:30 *** 11:30$(AM)

Name : ....

ID : ....

Exercise 1. Determine (up to an isomorphism) all Abelian groups of order 56.

Exercise 2. Let G be an Abelian group,

$$H = \{x^2 \mid x \in G\} \quad \text{and} \quad K = \{x \in G \mid x^2 = e\}.$$

(1) Show that H, K are subgroups of G.

(2) Show that $G/K \simeq H$.

Exercise 3. Let G_1, G_2 be two groups and H_1, H_2 be two normal subgroups of G_1, G_2 respectively.

(1) Show that $H_1 \oplus H_2$ is a normal subgroup of $G_1 \oplus G_2$.

(2) Show that $(G_1 \oplus G_2)/(H_1 \oplus H_2) \simeq (G_1/H_1) \oplus (G_2/H_2)$.

Exercise 4. Let m, n be positive integers and $f : \mathbb{Z}_n \longrightarrow \mathbb{Z}_m$ be a morphism of groups.

(1) Show that there exists $0 \leq a \leq m-1$ such that

$$na \equiv 0 \pmod m \text{ and } f(x + n\mathbb{Z}) = ax + m\mathbb{Z},$$

for all $x \in \mathbb{Z}$

(2) Show that the number of morphisms of groups from $\mathbb{Z}_n$ to $\mathbb{Z}_m$ is $d = \gcd(m, n)$.

Exercise 5. Let R be an integral domain, and

$$N : R - \{0\} \longrightarrow \mathbb{N} = \{0, 1, 2, \ldots\}$$

be a function satisfying $N(ab) = N(a)N(b)$, for all $a, b \in R - \{0\}$. Assume the set of all invertible elements of R is $U(R) = \{x \in R - \{0\} \mid N(x) = 1\}$.

(1) Show that if $x \in R - \{0\}$ and $N(x)$ is prime, then x is irreducible.

(2) Let $x \in R - \{0\}$ and y be a non-invertible divisor of x which is not associated with x (i.e., there is no $\varepsilon \in U(R)$ such that $y = \varepsilon x$). Suppose that $N(x) = p^2$, for some prime number p ; show that $p = N(y)$.

Exercise 6. Write $z = -33 + 9i$ as a product of irreducibles in $\mathbb{Z}[i]$.

Exercise 7. Let R be an integral domain.

(1) Show that if R is a PID, then every nonzero prime ideal of R is maximal.

(2) Show that $R[X]/(X.R[X]) \simeq R$. Deduce that $X.R[X]$ is a prime ideal of $R[X]$.

(3) Show that $R[X]$ is a PID if and only if R is a field.

Exercise 8. Let $R := \mathbb{Z}[i\sqrt{3}] = \{a + ib\sqrt{3} \mid a, b \in \mathbb{Z}\}$.

(1) Show that R is a subring of $\mathbb{C}$ with quotient field
$$\mathbb{Q}(i\sqrt{3}) := \{a + ib\sqrt{3} \mid a, b \in \mathbb{Q}\}.$$

(2) Show that $R \simeq \mathbb{Z}[X]/(X^2 + 3)\mathbb{Z}[X]$.

(3) Is $(X^2 + 3)\mathbb{Z}[X]$ a maximal ideal of $\mathbb{Z}[X]$?

(4) Find the set of all invertible elements of R.

(5) Show that $2, 1 + i\sqrt{3}, 1 - i\sqrt{3}$ are irreducible elements of R ; and deduce that R is not a UFD.