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King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 302 Final Exam
The Second Semester of 2016-2017 (162)

Time Allowed: 180 Minutes

Name: Key ID#: ______

Section/Instructor: ______ Serial #: ______

- Smart devices and calculators are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question #	Marks	Maximum Marks
1		16
2		14
3		10
4		16
5		16
6		15
7		17
8		18
9		18
Total		140

Q:1 (16 points) Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{pmatrix}$.

(a) Find the eigenvalues of A.
(b) Find the eigenvectors corresponding to the eigenvalues.
(c) Show that the eigenvectors are mutually orthogonal. Find an orthonormal set of eigenvectors.
(d) Construct matrices P and P^{-1} that orthogonally diagonalize the matrix A.

(a) $$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & -2 \\ 0 & -\lambda & 0 \\ -2 & 0 & 4-\lambda \end{vmatrix} = (1-\lambda)\left[\lambda^2-4\lambda\right]-2\left[-2\lambda\right]=0$$

$$\Rightarrow \lambda(1-\lambda)(\lambda-4)+4\lambda=0$$

$$\lambda(\lambda-\lambda^2+4\lambda-4+4)=0$$

or $$\lambda(-\lambda^2+5\lambda)=0 \Rightarrow \lambda=0,0,5$$

(b) For $\lambda=0$: $$(A-\lambda I \,\vdots\, 0) = \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 4 & 0 \end{array}\right)\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} k_1-2k_3=0 \\ -2k_1+4k_3=0 \end{array} \quad \Rightarrow k_1=2k_3$$

$$K_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda=5$:

$$(A-\lambda I : 0) = \left(\begin{array}{ccc|c} -4 & 0 & -2 & 0 \\ 0 & -5 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{array}\right)\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$\Rightarrow k_2=0 \quad \text{and} \quad k_3=-2k_1$$

$$K_3 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$$

(c) $K_1^T K_2 = 0$, $K_1^T K_2 = 0$, $K_2^T K_3 = 0$

$$\|K_1\| = 1, \quad \|K_2\| = \sqrt{5}, \quad \|K_3\| = \sqrt{5}$$

An orthonormal set of vectors is
$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2/\sqrt{5} \\ 0 \\ 1/\sqrt{5} \end{pmatrix}, \begin{pmatrix} -1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{pmatrix}$$

(d) Orthogonal matrix ($P^{-1} = P^T$)

$$P = \begin{pmatrix} 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

and $P^{-1} = P^T$

$$= \begin{pmatrix} 0 & 1 & 0 \\ \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \end{pmatrix}$$

Q:2 (14 points) Verify the divergence theorem (compute both sides) for the vector field

$$\widehat{F} = x\,\widehat{a}_x + y\,\widehat{a}_y + z\,\widehat{a}_z$$

over the sphere A of radius r centered at the origin.

Divergence theorem $\iint_S \hat{F}.\vec{n}\,dS = \iiint_V \nabla.\hat{F}\,dV$

(a) Divergence integral:

$$\nabla.\hat{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1+1+1 = 3$$

So $\iiint_A \nabla.\hat{F}\,dV = 3\iiint_A dV = 3 \cdot \frac{4}{3}\pi r^3 = 4\pi r^3$

(b) Flux integral:

Let $G(x,y,z) = x^2+y^2+z^2-r^2 = 0$

$$\nabla G = \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}, \quad |\vec{\nabla} G| = \sqrt{(2x)^2+(2y)^2+(2z)^2}$$

$$= 2r$$

and $\hat{n} = \frac{\vec{\nabla} G}{|\vec{\nabla} G|} = \frac{\langle 2x, 2y, 2z\rangle}{2r} = \frac{\langle x, y, z\rangle}{r}$

$$\hat{F}\cdot\hat{n} = \langle x, y, z\rangle \cdot \frac{\langle x, y, z\rangle}{r}$$

$$= \frac{x^2+y^2+z^2}{r} = \frac{r^2}{r} = r$$

$\therefore \iint_S \hat{F}.\hat{n}\,dS = \iint_S r\,dS$

$$= r \cdot 4\pi r^2$$

$$= 4\pi r^3$$

Q:3 (10 points) Verify that the function $u(x, y) = y^3 - 3y\,x^2 + x$ is harmonic in the entire plane. Find harmonic conjugate of $u(x, y)$.

Sol:
$$u_x = -6xy + 1 \quad ; \quad u_{xx} = -6y$$
$$u_y = 3y^2 - 3x^2 \quad ; \quad u_{yy} = 6y$$

$u_{xx} + u_{yy} = 0 \Rightarrow u$ is harmonic in the entire plane.

Now we have to find $v(x,y)$ for u such that the C.R. equ^ns^ are satisfied.

$$u_x = -6xy + 1 = v_y \qquad (1)$$
$$-u_y = 3x^2 - 3y^2 = v_x \qquad (2)$$

Integrating (1) w.r to y, we get
$$v(x,y) = -3xy^2 + y + g(x)$$
$$v_x(x,y) = -3y^2 + g'(x)$$
$$= 3x^2 - 3y^2 \quad (\text{by } (2))$$
$$\Rightarrow g'(x) = 3x^2$$
$$g(x) = x^3 + C$$
$$\therefore\ v(x,y) = -3xy^2 + x^3 + y + C.$$

Q:4 (6 + 5 + 5 points) Find the roots of $(1 + \sqrt{3}\, i)^{\frac{1}{4}}$.

$r = \sqrt{1+3} = 2, \quad \theta = \tan^{-1} \frac{\sqrt{3}}{1} = \frac{\pi}{3}$

$(1+\sqrt{3}\, i)^{1/4} = 2^{1/4}\left[\cos\left(\frac{\pi}{12} + \frac{k\pi}{2}\right) + i \sin\left(\frac{\pi}{12} + \frac{k\pi}{2}\right)\right]$

$k = 0, 1, 2, 3$

$w_0 = 2^{1/4}\left[\cos\frac{\pi}{12} + i \sin\frac{\pi}{12}\right]$

$w_1 = 2^{1/4}\left[\cos\frac{7\pi}{12} + i \sin\frac{7\pi}{12}\right]$

$w_2 = 2^{1/4}\left[\cos\frac{13\pi}{12} + i \sin\frac{13\pi}{12}\right]$

$w_3 = 2^{1/4}\left[\cos\frac{19\pi}{12} + i \sin\frac{19\pi}{12}\right]$

(b) Express $\cos(\frac{\pi}{2} + i \ln 2)$ in the form $a + ib$.

$\cos\left(\frac{\pi}{2} + i \ln 2\right) = \cos\frac{\pi}{2} \cos(i \ln 2) - \sin\frac{\pi}{2} \sin(i \ln 2)$

$= \cos\frac{\pi}{2} \cosh(\ln 2) - i \sin\frac{\pi}{2} \sinh(\ln 2)$

$= - i \sinh(\ln 2)$

$= - i \dfrac{e^{\ln 2} - e^{-\ln 2}}{2}$

$= - i \dfrac{2 - 1/2}{2} = - \dfrac{3}{4} i$

(c) Find all values of $(-i)^{4\,i}$.

$(-i)^{4i} = e^{4i \ln(-i)}$

$= e^{4i\left[\log_e|1| + i\left(-\frac{\pi}{2} + 2n\pi\right)\right]}$

$= e^{4i\left(i\left(-\pi/2 + 2n\pi\right)\right)}$

$= e^{2\pi - 8n\pi}, \quad n = 0, \pm 1, \pm 2, \ldots$

Q:5 (16 points) Let $f(z) = 4\,\bar{z} - i\,z$. Find the circulation around and net flux, around the circle $|z| = 2$.

Sol: The circulation and the net flux are the real part and imaginary part of $\oint_C \overline{f(z)}\,dz$.

$$\oint_C \overline{f(z)}\,dz = \oint_C (4z + i\bar{z})\,dz$$

Let $z = 2e^{i\theta}$, $0 \le \theta \le 2\pi$

$$\oint_C \overline{f(z)}\,dz = \int_0^{2\pi} \left[4 \cdot 2e^{i\theta} + i \cdot 2e^{-i\theta}\right] 2ie^{i\theta}\,d\theta$$

$$= 8i\int_0^{2\pi} e^{2i\theta}\,d\theta - 4\int_0^{2\pi} d\theta.$$

$$= \frac{8i}{2i}\left[e^{2i\theta}\right]_0^{2\pi} - 8\pi.$$

$$= 4\left(e^{4i\pi} - 1\right) - 8\pi$$

$$= 0 - 8\pi$$

Therefore

Circulation $= -8\pi$

Net flux $= 0$

Q:6 (15 points) Evaluate

$$\oint_C \frac{1}{(z-4)\,(z+1)^4}\,dz$$

where $C : |z-1| = \frac{5}{2}$.

Sol: We rewrite integral as

$$I = \oint_C \frac{\frac{1}{Z-4}}{(Z+1)^4}\,dZ$$

and notice that $f(Z) = \frac{1}{Z-4}$, $Z_0 = -1$ and $n = 3$.

Therefore

$$I = \frac{2\pi i}{3!} \left.\frac{d^3}{dZ^3}\left(\frac{1}{Z-4}\right)\right|_{Z=-1}$$

$$= \frac{2\pi i}{6}\left(\frac{-6}{-5^4}\right)$$

$$= -\frac{2\pi i}{5^4} = \frac{2\pi}{5^4 i}$$

$$\frac{d}{dZ}\left(\frac{1}{Z-4}\right) = -\frac{1}{(Z-4)^2}$$

$$\frac{d^2}{dZ^2}\left(\frac{1}{Z-4}\right) = -\frac{d}{dZ}\left(\frac{1}{(Z-4)^2}\right) = \frac{2}{(Z-4)^3}$$

$$\frac{d^3}{dZ^3}(\text{''}) = 2\frac{d}{dZ}\left(\frac{1}{(Z-4)^3}\right)$$

$$= \frac{-2.3(Z-2)^2}{(Z-4)^6} = -\frac{6}{(Z-4)^4}$$

Q:7 (9 + 8 points) (a) Expand

$$f(z) = \frac{1}{(z-1)^2(z-3)}$$

in a Laurent series valid for $0 < |z-1| < 2$.

We want to represent $f(z)$ in a series involving only powers of $z-1$. We want to express $z-3$ in terms of $z-1$.

$$f(z) = \frac{1}{(z-1)^2} \cdot \frac{1}{-2+(z-1)}$$

$$= -\frac{1}{2(z-1)^2} \cdot \frac{1}{1-\frac{z-1}{2}}$$

$$= -\frac{1}{2(z-1)^2} \underbrace{\left[1 + \frac{z-1}{2} + \frac{(z-1)^2}{2^2} + \cdots\right]}_{|z-1|<2}$$

$$= -\frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} - \frac{1}{8} - \frac{1}{16}(z-1) - \cdots$$

$$0 < |z-1| < 2$$

(b) Show that $z = 0$ is a removable singularity of $f(z) = \dfrac{\sin(4z) - 4z}{z^2}$.

$$f(z) = \frac{4z - (4z)^3/3! + (4z)^5/5! - \cdots - 4z}{z^2}$$

$$= -\frac{4^3 z}{3!} + \frac{4^5 z^3}{5!} - \frac{4^7 z^5}{7!} + \cdots$$

$\Rightarrow$ $z = 0$ is a removable singularity of $f(z)$.

Q:8 (18 points) Evaluate

$$\oint_{|z|=3} \frac{dz}{z^4 + z^3 - 2z^2}$$

by Cauchy's residue theorem

Sol:
$$f(Z) = \frac{1}{Z^4 + Z^3 - 2Z^2} = \frac{1}{Z^2(Z^2+Z-2)}$$
$$= \frac{1}{Z^2\,(Z+2)(Z-1)}$$

$f(Z)$ has simple poles at -2 and 1. It has a pole of order 2 at 0.

$$\operatorname{Res}(f(Z),0) = \lim_{Z\to 0} \frac{d}{dZ}\left[Z^2 f(Z)\right] = \lim_{Z\to 0} \frac{d}{dZ}\left[\frac{1}{(Z+2)(Z+1)}\right]$$
$$= \lim_{Z\to 0} \frac{-2Z-1}{(Z^2+Z-2)^2} = -\frac{1}{4}$$

$$\operatorname{Res}(f(Z),-2) = \lim_{Z\to -2} (Z+2) f(Z) = \lim_{Z\to -2} \frac{1}{Z^2(Z-1)} = -\frac{1}{12}$$

$$\operatorname{Res}(f(Z),1) = \lim_{Z\to 1} \frac{1}{Z^2(Z+2)} = \frac{1}{3}$$

Therefore, in view of the residue theorem, we obtain

$$\oint_{|Z|=3} \frac{dZ}{Z^4+Z^3-2Z^2} = 2\pi i\,[\text{sum of residues}]$$
$$= 2\pi i\left[-\frac{1}{4} - \frac{1}{12} + \frac{1}{3}\right]$$
$$= 0.$$

Q:9 (18 points) Evaluate

$$\int_0^{2\pi} \frac{1}{(3+2\cos\theta)^2}\, d\theta$$

by converting into a complex integral, where $C : |z| = 1$.

Let $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

$dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{z+\frac{1}{z}}{2}$

$$\int_C \frac{1}{(3+z+\frac{1}{z})^2}\frac{dz}{iz} = \frac{1}{i}\int_C \frac{z\,dz}{(z^2+3z+1)^2}$$

Here $F(z) = \frac{z}{(z^2+3z+1)^2} = \frac{z}{\left(z-\left(\frac{-3+\sqrt5}{2}\right)\right)^2\left(z-\left(\frac{-3-\sqrt5}{2}\right)\right)^2}$

$z_1 = -\frac{3+\sqrt5}{2}$, $z_2 = -\frac{3-\sqrt5}{2}$

Only z_1 is inside the unit circle C, Pole of order 2

$$\text{Res}(f(z), z_1) = \frac{1}{1!}\lim_{z\to z_1}\frac{d}{dz}\left[(z-z_1)^2 f(z)\right]$$

$$= \lim_{z\to z_1}\frac{d}{dz}\left[\frac{z}{(z-z_2)^2}\right]$$

$$= \lim_{z\to z_1}\frac{(z-z_2)^2 - 2z(z-z_2)}{(z-z_2)^4}$$

$$= \lim_{z\to z_1} -\frac{z_2 - z}{(z-z_2)^3} = -\frac{z_2-z_1}{(z_1-z_2)^3}$$

$$= \frac{(3+\sqrt5+3-\sqrt5)/2}{\left(\frac{-3+\sqrt5+3+\sqrt5}{2}\right)^3} = \frac{3}{(\sqrt5)^3} = \frac{3}{5\sqrt5}$$

$$\int_C = \frac{1}{i}\int_C = \frac{1}{i}\,2\pi i\,\text{Res}(f(z), z_1)$$

$$= 2\pi\cdot\frac{3}{5\sqrt5}$$

$$= \frac{6\pi}{5\sqrt5}$$