Key

001

King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 302 Major Exam II
The Second Semester of 2016-2017 (162)

Time Allowed: 120 Minutes

Name: ______________________ ID#: ______________

Section/Instructor: ______________ Serial #: ______________

- Mobiles and calculators are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question #	Marks	Maximum Marks
1		20
2		12
3		15
4		18
5		20
6		15
Total		100

Q:1 (20 points) Let $\widehat{A} = \rho\ \cos\phi\ \widehat{a}_\rho + z^2\ \sin\phi\ \widehat{a}_z$

(a) Transform $\widehat{A}$ into rectangular coordinates and calculate its magnitude at point $(3, -4, 0)$.

Sol(a):
$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \rho\cos\phi \\ 0 \\ z^2\sin\phi \end{bmatrix}$$

$$\rho^2 = x^2+y^2, \quad x = \rho\cos\phi,\ y = \rho\sin\phi$$

$$A_x = \rho\cos^2\phi = \sqrt{x^2+y^2}\ \frac{x^2}{x^2+y^2} = \frac{x^2}{\sqrt{x^2+y^2}}$$

$$A_y = \rho\sin\phi\cos\phi = \sqrt{x^2+y^2}\ \frac{xy}{x^2+y^2} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$A_z = z^2\sin\phi = z^2\frac{y}{\sqrt{x^2+y^2}} = \frac{yz^2}{\sqrt{x^2+y^2}}$$

$$\hat{A} = A_x\hat{a}_x + A_y\hat{a}_y + A_z\hat{a}_z$$
$$= \frac{x^2}{\sqrt{x^2+y^2}}\hat{a}_x + \frac{xy}{\sqrt{x^2+y^2}}\hat{a}_y + \frac{yz^2}{\sqrt{x^2+y^2}}\hat{a}_z$$
$$= \frac{9}{5}\hat{a}_x - \frac{12}{5}\hat{a}_y + 0\,\hat{a}_z$$

$$|\hat{A}| = \sqrt{\frac{81+144}{25}} = \sqrt{\frac{225}{25}}$$
$$= \sqrt{9}$$
$$= 3$$

(b) Transform $\hat{A}$ into spherical coordinates.

Sol:
$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} x^2/\rho \\ xy/\rho \\ yz^2/\rho \end{bmatrix}$$

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \qquad \rho = r\sin\theta \end{aligned}}$$

$$A_r = \frac{r^2\sin^2\theta\cos^2\phi}{r\sin\theta}\sin\theta\cos\phi + \frac{r^2\sin^2\theta\cos\phi\sin\phi}{r\sin\theta}\sin\theta\sin\phi + \frac{r^3\sin\theta\sin\phi\cos^2\theta}{r\sin\theta}\cdot\cos\theta$$

$$= r\sin^2\theta\cos^3\phi + r\sin^2\theta\cos\phi\sin^2\phi + r^2\sin\phi\cos^3\theta$$

$$= r\sin^2\theta\cos\phi(\cos^2\phi + \sin^2\phi) + r^2\sin\phi\cos^3\theta$$

$$= r\sin^2\theta\cos\phi + r^2\sin\phi\cos^3\theta$$

$$A_\theta = \frac{r^2\sin^2\theta\cos^2\phi}{r\sin\theta}\cos\theta\cos\phi + \frac{r^2\sin^2\theta\cos\phi\sin\phi}{r\sin\theta}\cos\theta\sin\phi - \frac{r^3\sin\theta\sin\phi\cos\theta}{r\sin\theta}\sin\theta$$

$$= r\sin\theta\cos^3\phi\cos\theta + r\sin\theta\cos^2\phi\sin^2\phi - r^2\sin\phi\cos\theta\sin\theta$$

$$A_\phi = -\frac{r^2\sin^2\theta\cos^2\phi}{r\sin\theta}\sin\phi + \frac{r^2\sin^2\theta\cos\phi\sin\phi}{r\sin\theta}\cos\phi + 0$$

$$= -r\sin\theta\cos^2\phi\sin\phi + r\sin\theta\cos^2\phi\sin\phi = 0$$

OR

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta & 0 & \cos\theta \\ \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \rho\cos\phi \\ 0 \\ z^2\sin\phi \end{bmatrix}$$

$$\Rightarrow \quad A_r = r\sin^2\theta\cos\phi + r^2\cos^3\theta\sin\phi$$

$$A_\theta = r\sin\theta\cos\phi\cos\theta - r^2\sin\phi\cos^2\theta\sin\theta$$

$$A_\phi = 0$$

Q:2 (12 points) (a) Calculate the distance between the points $(3, \frac{\pi}{2}, -1)$ and $(5, \frac{3\pi}{2}, 5)$ in cylindrical coordinates.

Sol: $d^2 = \rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2$

$= 9 + 25 - 30\cos\pi + (5+1)^2$

$= 34 + 36 - 30\cos\pi$

$= 70 + 30$

$= 100$

$\Rightarrow d = 10$

(b) Find $\nabla^2 w$, where $W = 10\, r \sin^2\theta \cos\phi$ in spherical coordinates.

Sol:

$\nabla^2 W = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial W}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial W}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 W}{\partial\phi^2}$

$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(10 r^2 \sin^2\theta \cos\phi\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \cdot 10 r \cdot 2\sin\theta \cos\theta \cos\phi\right)$

$+ \frac{1}{r^2\sin^2\theta}\left(-10 r \sin^2\theta \cos\phi\right)$

$= \frac{1}{r^2}\cdot 20 r \sin^2\theta \cos\phi + \frac{1}{r^2\sin\theta}\left(10 r \cos\phi \left(\sin 2\theta \cos\theta + 2\cos 2\theta \sin\theta\right)\right)$

$- \frac{10}{r}\cos\phi$

Q:3 (15 points) Let $f(x, y, z) = xy\, e^{x^2+z^2-5}$. Calculate the directional derivative at the point $(1, 3, -2)$ in the direction $\widehat{V} = (3, -1, 4)$ (Using the gradient).

Solution: We have

$$\hat{\nabla} f(x,y,z) = \left\langle (y + 2x^2 y)e^{x^2+z^2-5},\ x e^{x^2+z^2-5},\ 2xyz\, e^{x^2+z^2-5} \right\rangle$$

$$\hat{\nabla} f(1,3,-2) = \langle 9, 1, -12 \rangle$$

We consider the unit vector

$$\hat{u} = \frac{\hat{V}}{\|\hat{V}\|} = \left\langle \frac{3}{\sqrt{26}}, \frac{-1}{\sqrt{26}}, \frac{4}{\sqrt{26}} \right\rangle.$$

The directional derivative is given by

$$D_{\hat{u}} f(1,3,-2) = \hat{\nabla} f(1,3,-2) \cdot \hat{u}$$

$$= \langle 9, 1, -12 \rangle \cdot \left\langle \frac{3}{\sqrt{26}}, \frac{-1}{\sqrt{26}}, \frac{4}{\sqrt{26}} \right\rangle$$

$$= \frac{27 - 1 - 48}{\sqrt{26}}$$

$$= \frac{-22}{\sqrt{26}}$$

y^2

Q:4 (18 points) Use Stokes's Theorem to evaluate $\oint_C (x+z)dx + (x-y)\,dy + x\,dz$, where C is the ellipse defined by $\frac{x^2}{4} + \frac{x^2}{9} = 1, \quad z = 1$.

The curl of vector field is

$$\nabla \times \hat{F} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+z & x-y & x \end{vmatrix}$$

$$= (0-0)\hat{a}_x - \hat{a}_y(1-1) + (1-0)\hat{a}_z$$

$$= \hat{a}_z$$

Stokes's theorem gives (where $d\hat{S} = \hat{a}_z dS$)

$$\int_C \hat{F}\cdot d\hat{l} = \iint_S (\nabla \times \hat{F})\cdot d\hat{S}$$

$$= \iint_S \hat{a}_z \cdot \hat{a}_z \, dS$$

$$= \iint_S dS$$

$$= \pi \cdot 2 \cdot 3$$

$$= 6\pi$$

Q:5 (20 points) Let $\widehat{E} = 4\ \rho\ \sin\phi\ \widehat{a}_\rho + 2\ \rho\ \cos\phi\ \widehat{a}_\phi + 2\ z^2\ \widehat{a}_z$ be the electric field on a certain region of space.

(a) Verify that $\widehat{E}$ is a conservative field.

(b) Find the electric potential V.

(c) Use the Fundamental theorem and potential function to evaluate $\int_{(1,0,-2)}^{(4,\frac{\pi}{2},1)} \widehat{E}.\widehat{dl}$.

(a)

$$\nabla \times \hat{E} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\,\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 4\rho\sin\phi & \rho\cdot 2\rho\cos\phi & 2z^2 \end{vmatrix}$$

$$= \frac{1}{\rho}\left[(0-0)\,\hat{a}_\rho - (0-0)\,\rho\hat{a}_\phi + (4\rho\cos\phi - 4\rho\cos\phi)\hat{a}_z\right]$$

$$= \hat{0}$$

$\Rightarrow$ E is conservative field

(b) $\hat{E} = -\nabla V \iff \hat{E} = -\left\langle \frac{\partial V}{\partial \rho}, \frac{1}{\rho}\frac{\partial V}{\partial\phi}, \frac{\partial V}{\partial z}\right\rangle$

$\Rightarrow \frac{\partial V}{\partial\rho} = -4\rho\sin\phi \quad ; \quad \frac{1}{\rho}\frac{\partial V}{\partial\phi} = -2\rho\cos\phi \quad ; \quad \frac{\partial V}{\partial z} = -2z^2$

$V = -2\rho^2\sin\phi + G(\phi, z)$

$\frac{\partial V}{\partial \phi} = -2\rho^2\cos\phi + G_\phi(\phi, z) = -2\rho^2\cos\phi$

$\Rightarrow G_\phi(\phi, z) = 0 \quad \Rightarrow G(\phi, z) = H(z)$

Therefore

$V = -2\rho^2\sin\phi + H(z)$

$\frac{\partial V}{\partial z} = 0 + H'(z) = -2z^2 \quad \Rightarrow H(z) = -\frac{2}{3}z^3 + C$

Hence $V = -2\rho^2\sin\phi - \frac{2}{3}z^3 + C$

(c) $V(1,0,-2) = -2\cdot 1\cdot 0 - \frac{2}{3}(-8) = \frac{16}{3}$

$V(4, \frac{\pi}{2}, 1) = -2(16)\cdot 1 - \frac{2}{3}\cdot 1 = -32 - \frac{2}{3} = -\frac{98}{3}$

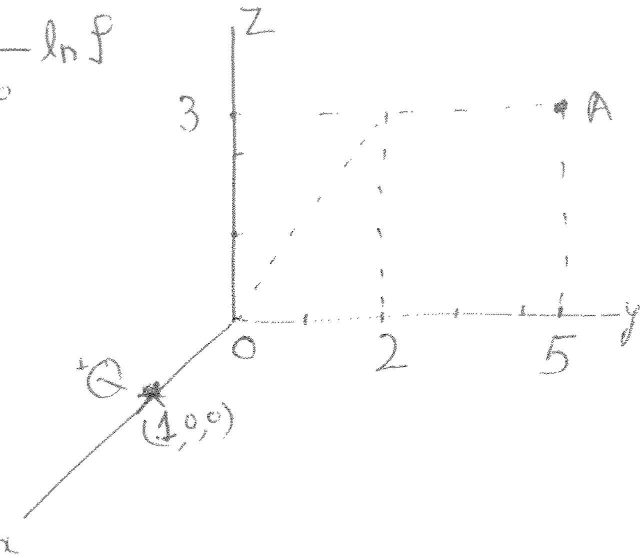
By Fundamental theorem

$$\int_{(1,0,-2)}^{(4,\frac{\pi}{2},1)} \hat{E}\cdot\hat{dl} = V(4, \pi/2, 1) - V(1,0,-2)$$

$$= -\frac{98}{3} - \frac{16}{3} = -\frac{114}{3}$$

Q:6 (15 points) (a) A point charge of 8 nC is located at $(1, 0, 0)$, while line $y = 2, z = 3$ carries a uniform charge of 3 nC. If the potential at $A(0, 5, 3)$ is 30V, what is the potential at $(0, 0, 0)$?

$$V = V_Q + V_L = \frac{Q}{4\pi\epsilon_0 r} - \frac{\rho_L}{2\pi\epsilon_0}\ln\rho$$



$$r_0 = |(0,0,0) - (1,0,0)| = 1$$

$$\rho_0 = |(0,0,0) - (0,2,3)| = \sqrt{2^2+3^2} = \sqrt{13}$$

$$r_A = |(0,5,3) - (1,0,0)| = \sqrt{1+25+9} = \sqrt{35}$$

$$\rho_A = |(0,5,3) - (0,2,3)| = 3$$

Hence

$$V_0 - V_A = -\frac{\rho_L}{2\pi\epsilon_0}\ln\frac{\rho_0}{\rho_A} + \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r_0} - \frac{1}{r_A}\right]$$

$$= \frac{1}{2\pi\epsilon_0}\left[-3\ln\frac{\sqrt{13}}{3} + \frac{8}{2}\left\{1 - \frac{1}{\sqrt{35}}\right\}\right]$$

As $V_A = 30$, we obtain

$$V_0 = 30 + \frac{1}{2\pi\epsilon_0}\left\{-3\ln\frac{\sqrt{13}}{3} + 4\left(\frac{35-\sqrt{35}}{35}\right)\right\}$$

(b) Calculate the work done in moving a 5 μC charge from $A(2, \frac{\pi}{4}, 2)$ to $B(3, \frac{\pi}{3}, 4)$ given the potential $V = \frac{z^3}{\rho}\sin\phi$.

$$W = -Q\int_L \hat{E}\cdot d\hat{l} = +5[V(B) - V(A)]$$

$$= 5\left[\frac{4^3}{3}\sin\frac{\pi}{3} - \frac{2^3}{2}\sin\frac{\pi}{4}\right].$$

$$= 5\left[\frac{32}{3}\cdot\sqrt{3} - \frac{4}{\sqrt{2}}\right]$$

$$= \frac{10}{3}\left[16\sqrt{3} - 3\sqrt{2}\right]$$

Formulae in cylindrical and spherical coordinate systems

Differential of displacement

Cylindrical: $d\boldsymbol{l} = d\rho\, \widehat{\boldsymbol{a}_\rho} + \rho d\phi\, \widehat{\boldsymbol{a}_\phi} + dz\, \widehat{\boldsymbol{a}_z}$

Spherical: $d\boldsymbol{l} = dr\, \widehat{\boldsymbol{a}_r} + r d\theta\, \widehat{\boldsymbol{a}_\theta} + r\sin\theta\, d\phi\, \widehat{\boldsymbol{a}_\phi}$

Gradient of a scalar field, ∇ V

Cylindrical: $\nabla \mathrm{V} = \frac{\partial V}{\partial \rho}\widehat{\boldsymbol{a}_\rho} + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\widehat{\boldsymbol{a}_\phi} + \frac{\partial V}{\partial z}\widehat{\boldsymbol{a}_z} = < \frac{\partial V}{\partial \rho}, \frac{1}{\rho}\frac{\partial V}{\partial \phi}, \frac{\partial V}{\partial z} >$

Spherical: $\nabla \mathrm{V} = \frac{\partial V}{\partial r}\widehat{\boldsymbol{a}_r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\widehat{\boldsymbol{a}_\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\widehat{\boldsymbol{a}_\phi} = < \frac{\partial V}{\partial r}, \frac{1}{r}\frac{\partial V}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} >$

Divergence of a vector field, $\nabla \cdot \mathbf{G}$

Cylindrical: $\nabla \cdot \mathbf{G} = \frac{1}{\rho}\frac{\partial(\rho G_\rho)}{\partial \rho} + \frac{1}{\rho}\frac{\partial(G_\phi)}{\partial \phi} + \frac{\partial(G_z)}{\partial z}$, where $G =< G_\rho, G_\phi, G_z >$

Spherical: $\nabla \cdot \mathbf{G} = \frac{1}{r^2}\frac{\partial(r^2 G_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, G_\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial(G_\phi)}{\partial \phi}$,

where $G =< G_r, G_\theta, G_\phi >$

Relationship between Cartesian, Cylindrical and Spherical Coordinates

$$\begin{pmatrix} A_\rho \\ A_\varphi \\ A_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\varphi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

Differential of normal surface

Cylindrical:
$$d\boldsymbol{S} = \rho d\phi dz\ \widehat{\boldsymbol{a}_\rho}$$
$$d\rho dz\ \widehat{\boldsymbol{a}_\phi}$$
$$\rho d\rho d\phi\ \widehat{\boldsymbol{a}_z}$$

Spherical:
$$d\boldsymbol{S} = r^2 \sin\theta d\theta\, d\phi\ \widehat{\boldsymbol{a}_r}$$
$$r\sin\theta dr\, d\phi\ \widehat{\boldsymbol{a}_\theta}$$
$$r dr d\theta\ \widehat{\boldsymbol{a}_\phi}$$

Curl of a vector field, $\nabla \times \mathbf{G}$

Cylindrical:
$$\nabla \times \mathbf{G} = \frac{1}{\rho}\begin{vmatrix} \widehat{\boldsymbol{a}_\rho} & \rho\,\widehat{\boldsymbol{a}_\phi} & \widehat{\boldsymbol{a}_z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ G_\rho & \rho G_\phi & G_z \end{vmatrix},$$

where $G =< G_\rho, G_\phi, G_z >$

Spherical:
$$\nabla \times \mathbf{G} = \frac{1}{r^2 sin\theta}\begin{vmatrix} \widehat{\boldsymbol{a}_r} & r\,\widehat{\boldsymbol{a}_\theta} & rsin\theta\widehat{\boldsymbol{a}_\phi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ G_r & rG_\theta & rsin\theta G_\phi \end{vmatrix},$$

where $G =< G_r, G_\theta, G_\phi >$

Laplacian of a scalar field, $\nabla^2 V$

Cylindrical: $\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

Spherical: $\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$