Key

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King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 302 Major Exam I
The Second Semester of 2016-2017 (162)

Time Allowed: 120 Minutes

Name: ____________________ ID#: ____________

Section/Instructor: ______________ Serial #: ____________

- Mobiles and calculators are not allowed in this exam.
- Write neatly and eligibly. You may lose points for messy work.
- Show all your work. No points for answers without justification.

Question #	Marks	Maximum Marks
1		24
2		10
3		10
4		20
5		20
6		16
Total		100

Q:1 (24 points) Let $E = \{(x, y, z) \in R^3 : x + y - 2z = 0 \ and \ 2x - y - z = 0\}$

and $F = \{(x, y, z) \in R^3 : x + y - z = 0\}$.

Let also $a = (1, 1, 1), b = (1, 0, 1) \ and \ c = (0, 1, 1)$.

(a) Prove that E is a subspace of R^3.

Notice that $\begin{array}{l} 0 + 0 - 2 \times 0 = 0 \\ 2 \times 0 - 0 - 0 = 0 \end{array} \Rightarrow 0 \in E$

Let $u_1 = (x_1, y_1, z_1) \in E$, $u_2 = (x_2, y_2, z_2) \in E$.

Therefore

$\begin{array}{l} x_1 + y_1 - 2z_1 = 0 \\ 2x_1 - y_1 - z_1 = 0 \end{array}$ and $\begin{array}{l} x_2 + y_2 - 2z_2 = 0 \\ 2x_2 + y_2 - z_2 = 0. \end{array}$

We have for $\lambda_1, \lambda_2 \in R$,

$$\lambda_1 u_1 + \lambda_2 u_2 = (\lambda_1 x_1 + \lambda_2 x_2, \lambda_1 y_1 + \lambda_2 y_2, \lambda_1 z_1 + \lambda_2 z_2)$$

Now,

$$(\lambda_1 x_1 + \lambda_2 x_2) + (\lambda_1 y_1 + \lambda_2 y_2) - 2(\lambda_1 z_1 + \lambda_2 z_2)$$
$$= \lambda_1 \underbrace{(x_1 + y_1 - 2z_1)}_{0} + \lambda_2 \underbrace{(x_2 + y_2 - 2z_2)}_{0}$$
$$= 0$$

and

$$2(\lambda_1 x_1 + \lambda_2 x_2) - (\lambda_1 y_1 + \lambda_2 y_2) - (\lambda_1 z_1 + \lambda_2 z_2)$$
$$= \lambda_1 \underbrace{(2x_1 - y_1 - z_1)}_{0} + \lambda_2 \underbrace{(2x_2 - y_2 - z_2)}_{0} = 0$$

Therefore $\lambda_1 u_1 + \lambda_2 u_2 \in E$ i.e E is closed with respect to both: addition and multiplication by a real number.

(b) Find a family which spans E and prove that this family is a basis.

Combining the 2 conditions in the definition of E:

$$x + y - 2z = 0 \quad \text{and} \quad 2x - y - z = 0$$

$$\Rightarrow \quad y = 2z - x = 2x - z$$

$$\Rightarrow x = z$$

and $y = 2x - x = x \quad \Rightarrow x = y = z$

Hence $\{a\}$ spans E and it is a basis for E.

(c) Prove that $\{b, c\}$ is a basis for F.

We have $1 + 0 - 1 = 0 \Rightarrow b \in F$

$0 + 1 - 1 = 0 \Rightarrow c \in F$

b and c are not proportional i.e are not multiple of each other**. Therefore, they are linearly independent.

Notice also that

$(1,0,0) \notin F$. Hence $F \subsetneq \mathbb{R}^3$ and we

Conclude that $\dim(F) = 2$ and $\{b, c\}$ is a basis.

** [Assume, on the contrary that $b = \alpha c$, for some $\alpha \neq 0$,
then $(1,0,1) = \alpha(0,1,1) = (0,\alpha,\alpha) \Rightarrow 1 = 0$ (Impossible)
A similar argument shows that c is not multiple of b.

Q:2 (10 points) Solve the following system using **Gaussian Elimination method:**

$$2x_1 - 4x_2 + 3x_3 = 0$$

$$x_1 + x_2 - 2x_3 = 0$$

$$\left(\begin{array}{ccc|c} 2 & -4 & 3 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & -4 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -6 & 7 & 0 \end{array}\right) \xrightarrow{-\frac{R_2}{6}} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 1 & -\frac{7}{6} & 0 \end{array}\right)$$

$$\Rightarrow \quad x_1 + x_2 - 2x_3 = 0$$
$$x_2 - \frac{7}{6}x_3 = 0$$

Let $x_3 = t$. Then $x_2 = \frac{7}{6}t$, $x_1 = \frac{5}{6}t$.

For $t = 0$, Solution is trivial and for $t \neq 0$, Solution is nontrivial

Q:3 (10 points) Let A be a non-zero 7×9 matrix.

(a) What is the maximum rank that A can have?

The maximum possible rank of A is the number of rows in A, which is 7.

(b) If $\text{rank}(A|B) = 6$, then for what value(s) of rank A is the system $AX = B$, $B \neq 0$ inconsistent? consistent?

The system is inconsistent if $\text{rank}(A) < 6$.

The " " consistent if $\text{rank}(A) = \text{rank}(A|B) = 6$

(C) If rank $(A) = 3$, then how many parameters does the solution of the system $AX = 0$ have?

$n = 9$ unknowns and the rank of A is $r = 3$.

Thus, the Solution of the System has $n - r = 9 - 3 = 6$ parameters

Q:4 (20 points) (a) Use Gauss-Jordan elimination method to find matrix A if $A^{-1} = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix}$.

To Find A, we compute $(A^{-1})^{-1} = A$.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & -1 & 2 & 0 & 1 & 0 \\ -2 & 2 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3+2R_1]{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 3 & -2 & -2 & 1 & 0 \\ 0 & -2 & 3 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2+R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -2 & 3 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_3+2R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 6 & 5 & 2 & 2 & 3 \end{array}\right)$$

$$\xrightarrow{\frac{R_3}{5}} \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & \frac{2}{5} & \frac{2}{5} & \frac{3}{5} \end{array}\right) \xrightarrow[R_2-R_3]{R_1+2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 2 & 2 \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{2}{5} & \frac{2}{5} & \frac{3}{5} \end{array}\right)$$

$$\xrightarrow{R_1-4R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -\frac{2}{5} & \frac{3}{5} & \frac{2}{5} \\ 0 & 0 & 1 & \frac{2}{5} & 2/5 & 3/5 \end{array}\right).$$

We have $A = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} & -\frac{2}{5} \\ -2/5 & 3/5 & 2/5 \\ 2/5 & 2/5 & 3/5 \end{pmatrix}$

(b) Solve the system $\mathbf{AX} = \mathbf{B}$, where A is the matrix found in (a),

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } B = \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix}.$$

$$X = A^{-1} B = \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ -2 & 2 & -1 \end{pmatrix} \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix} = \begin{pmatrix} 15 \\ 30 \\ -5 \end{pmatrix}$$

Q:5 (20 points) (a) Is $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ an orthogonal matrix ?

$$|M| = \cos^2\theta + \sin^2\theta \neq 0$$

$$M^T = M^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$\Rightarrow$ M is an orthogonal matrix.

(b) Let $A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The matrix A has $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = -2$ as eigenvalues. For λ_1, we have 2 corresponding eigenvectors

$K_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $K_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. and for λ_3 we have $K_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$.

(i) Is $\{K_1, K_2, K_3\}$ an orthogonal set ? If not, find an orthogonal set of eigenvectors.

(ii) Find an orthonormal set of eigenvectors.

(iii) Find a matrix Q such that

$$A = Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} Q^{-1}.$$

Hint: Eigenvectors corresponding to λ_1 have the form $\begin{pmatrix} v-u \\ u \\ v \end{pmatrix}$.

$$\boxed{\lambda_3 = -2}$$

Cont.

b(i) $\{K_1, K_2, K_3\}$ is <u>not</u> an orthogonal set because K_1 and K_2 are not orthogonal.

We will replace K_2 by another $\tilde{K}_2$ which is still an eigenvector but is orthogonal to K_1.

We know that the eigenvectors corresponding to λ_1 have the form $\begin{pmatrix} v-u \\ u \\ v \end{pmatrix}$ (from the hint).

So $\begin{pmatrix} v-u \\ u \\ v \end{pmatrix}^T \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0 \Rightarrow -v+u+u=0$
$\Rightarrow v = 2u$

Therefore, we may pick $\tilde{K}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

(ii) Normalizing the eigenvectors

$$\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

(iii) Clearly

$$Q = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \end{pmatrix}$$

Q:6 (16 points) (i) Is $A = \begin{pmatrix} 1 & \alpha & \beta \\ 0 & 2 & \gamma \\ 0 & 0 & 3 \end{pmatrix}$ diagonalizable (here $\alpha, \beta, \gamma \in R$)?

Observe that A is upper triangular matrix. Therefore the eigenvalues are the diagonal elements $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$.

The eigenvalues are distinct, therefore it is diagonalizable

(ii) Let $a, b \in R$ and $M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

Is M invertible ? Is M diagonalizable ? If yes, find the diagonal matrix.

observe that M is symmetric. $|M| = a^2 - b^2 = 0 \Leftrightarrow |a| = |b|$.

Therefore M is invertible if and only if $|a| \neq |b|$

We have

$$P(\lambda) = (a-\lambda)^2 - b^2$$
$$= (a+b-\lambda)(a-b-\lambda)$$

If $b \neq 0$, then $\lambda_1 = a+b \neq a-b = \lambda_2$

$\Rightarrow$ M is diagonalizable and $\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$

If $b = 0$, then $M = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ is clearly diagonal matrix.

Conclusion: In all cases $(\forall a, \forall b)$

M is similar to $\begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$.