King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 301 - Final Exam - Term 162

Duration: 180 minutes

Name: Key ID Number:

Section Number: ______ Serial Number: ______

Class Time: ______ Instructor's Name: ______

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have total of 8 Questions.

Question Number	Points	Maximum Points
1		12
2		15
3		20
4		15
5		20
6		20
7		20
8		18
Total		140

1. **[12 points]** Consider the Sturm-Lioville problem:

$$x^2y'' + xy' + 9\lambda\, y = 0, \qquad 1 < x < e,$$

$$y(1) = 0, \quad y(e) = 0,$$

where the eigenvalues $\lambda > 0$.

a) **[6 points]** Find the eigenvalues $\lambda > 0$ and corresponding eigenfunctions of the above problem.

b) **[3 points]** Put the differential equation in self-adjoint form.

c) **[3 points]** Give an orthogonality relation.

Ans: a) Let $\lambda = \alpha^2, \ \alpha > 0$.

DE: $x^2y'' + xy' + 9\lambda y = 0$ has AE: $m(m-1) + m + 9\alpha^2 = 0$

$\Rightarrow m = \pm 3\alpha i \Rightarrow y = c_1 \cos(3\alpha \ln x) + c_2 \sin(3\alpha \ln x)$ $\boxed{2\text{pts}}$

BC: $y(1) = 0$: $\boxed{0 = c_1}$

$y(e) = 0$: $0 = c_2 \sin 3\alpha \Rightarrow 3\alpha = n\pi, \ n = 1, 2, \dots \Rightarrow \alpha_n = \frac{n\pi}{3}$

Eigenvalues: $\lambda_n = \left(\frac{n\pi}{3}\right)^2, \ n = 1, 2, \dots$ $\boxed{2\text{pts}}$

Eigenfunctions: $y_n = \sin(3\alpha_n \ln x) \Rightarrow y_n = \sin(n\pi \ln x)$. $\boxed{2\text{pts}}$

b) DE: $y'' + \frac{1}{x}y' + \frac{9\lambda}{x^2}y = 0$. Int. fac $= e^{\int \frac{1}{x}dx} = x$

Then $xy'' + y' + \frac{9\lambda}{x}y = 0$. $\boxed{2\text{pts}}$

Self-adjoint form: $(xy')' + \frac{9\lambda}{x}y = 0$. $\boxed{1\text{pt}}$

c) $p(x) = \frac{9}{x}$.

Orthogonality relation: $(y_n, y_m) = \int_1^e \frac{9}{x} y_n y_m \, dx = 0, \quad n \neq m$ $\boxed{2\text{pts}}$

$(y_n, y_m) = \int_1^e \frac{9}{x} \sin(n\pi \ln x)\sin(m\pi \ln x)\, dx$ $\boxed{1\text{pt}}$

$n \neq m$.

2. **[15 points]** Write the first three nonzero terms of the Fourier Legendre series of

$$f(x) = |x|, \qquad -1 < x < 1.$$

(Simplify your final answer as a polynomial of degree 4.)

Ans: $f(x) = \sum_{n=0}^{\infty} c_n P_n(x), \qquad c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x) P_n(x)\,dx$ $\boxed{1\text{pt}}$

Since f is an even function, $c_1 = c_3 = \cdots = 0$ $\boxed{2\text{pts}}$

$$c_0 = \frac{1}{2}\int_{-1}^{1} |x|\,dx = \int_0^1 x\,dx = \frac{1}{2} \qquad \boxed{3\text{ pts}}$$

$$c_2 = \frac{5}{2}\int_{-1}^{1} |x| \cdot \frac{1}{2}(3x^2-1)\,dx = \frac{5}{2}\int_0^1 (3x^3 - x)\,dx$$

$$= \frac{5}{2}\left(\frac{3}{4}x^4 - \frac{1}{2}x^2\right)\Big|_0^1 = \frac{5}{8}. \qquad \boxed{3\text{pts}}$$

$$c_4 = \frac{9}{2}\int_{-1}^{1} |x| \cdot \frac{1}{8}(35x^4 - 30x^2 + 3)\,dx = \frac{9}{8}\int_0^1 (35x^5 - 30x^3 + 3x)\,dx$$

$$= \frac{9}{8}\left(\frac{35}{6} - \frac{30}{4} + \frac{3}{2}\right) = \frac{-3}{16} \qquad \boxed{3\text{ pts}}$$

Then, $f(x) = |x| = c_0 P_0(x) + c_2 P_2(x) + c_4 P_4(x) + \cdots$

$$= \frac{1}{2} + \frac{5}{16}(3x^2 - 1) - \frac{3}{128}(35x^4 - 30x^2 + 3) + \cdots \qquad \boxed{3\text{ pts}}$$

$$= -\frac{105}{128}x^4 + \frac{105}{64}x^2 + \frac{15}{128} + \cdots$$

3. **[20 points]** Use separation of variables to solve the initial boundary value problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad 0 < x < 1,$$

$$u(0,t) = 0, \quad u(1,t) = 0, \qquad t > 0,$$

$$u(x,0) = 2\sin 3\pi x, \qquad 0 < x < 1.$$

Ans: Let $u(x,t) = X(x)\,T(t)$ $\boxed{2\text{pts}}$

DE: $X''T = XT' \Rightarrow \dfrac{X''}{X} = \dfrac{T'}{T} = -\lambda$ $\boxed{2\text{pts}}$

Then, we have ① $X'' + \lambda X = 0$ and ② $T' + \lambda T = 0$

① $X'' + \lambda X = 0$

BC: $X(0) = 0$ and $X(1) = 0$

Case 1: $\lambda = 0$

$X = c_1 + c_2 x$

$X(0) = 0$: $c_1 = 0$ $\Rightarrow X = 0.$ $\boxed{2\text{pts}}$

$X(1) = 0$: $c_2 = 0$

Case 2: $\lambda = -\alpha^2$, $\alpha > 0$

$X = c_1 \cosh \alpha x + c_2 \sinh \alpha x$

$X(0) = 0$: $c_1 = 0$

$X(1) = 0$: $0 = c_2 \sinh \alpha$

Since $\sinh \alpha \neq 0$, $c_2 = 0$

Hence, $X = 0.$ $\boxed{2\text{pts}}$

Case 3: $\lambda = \alpha^2$, $\alpha > 0$

$X = c_1 \cos \alpha x + c_2 \sin \alpha x$

Hence, $\boxed{\lambda_n = (n\pi)^2}$, $n = 1, 2, \ldots$

$\boxed{X_n = \sin(n\pi x)}$. $\boxed{4\text{ pts}}$

② $T' + (n\pi)^2 T = 0$ $\boxed{2\text{pts}}$

$\Rightarrow T = C\, e^{-(n\pi)^2 t}$.

That is, $T_n = e^{-(n\pi)^2 t}$

Now, $u(x,t) = \sum_{n=1}^{\infty} A_n X_n T_n$

$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi)^2 t} \sin(n\pi x)$

IC: $u(x,0) = 2\sin 3\pi x$ gives $\boxed{2}$

$2\sin 3\pi x = \sum_{n=1}^{\infty} A_n \sin n\pi x$

$\Rightarrow$ For $n = 3$, $A_3 = 2$ $\boxed{2\text{pts}}$

For $n \neq 3$, $A_n = 0$.

$\Rightarrow u(x,t) = 2\, e^{-9\pi^2 t} \sin 3\pi x$

$\boxed{2\text{pts}}$

4. a) [~~1~~5 **points**] Find constants A_n such that $u_1(x,y) = \sum_{n=1}^{\infty} A_n \sinh ny \sin nx$ is the solution of the problem:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0, \qquad x, y \in (0,\pi),$$

$$u_1(0,y) = 0, \quad u_1(\pi,y) = 0, \quad y \in (0,\pi),$$

$$u_1(x,0) = 0, \quad u_1(x,\pi) = 1, \quad x \in (0,\pi).$$

Hint: Use the nonhomogenous boundary condition: $u_1(x,\pi) = 1$.

Ans: $u_1(x,y) = \sum_{n=1}^{\infty} A_n \sinh ny \sin nx$ satisfies $u_1(x,\pi) = 1$.

Then, $1 = \sum_{n=1}^{\infty} A_n \sinh n\pi \sin nx$ $\boxed{1\text{pt}}$

$\Rightarrow A_n \sinh n\pi = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin nx \, dx$ $\boxed{2\text{pts}}$

$\Rightarrow A_n = \frac{2(1-(-1)^n)}{n\pi \sinh n\pi}$ $\boxed{2\text{pts}}$

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b) [**5 points**] Find constants B_n such that $u_2(x,y) = \sum_{n=1}^{\infty} B_n \sinh nx \sin ny$ is the solution of the problem:

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0, \quad x, y \in (0,\pi),$$

$$u_2(0,y) = 0, \quad u_2(\pi, y) = 1, \quad y \in (0,\pi),$$

$$u_2(x,0) = 0, \quad u_2(x,\pi) = 0, \quad x \in (0,\pi).$$

Ans: Similar to part a),

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \sinh nx \sin ny$$ satisfies

$u_2(\pi, y) = 1$. Then,

$$1 = \sum_{n=1}^{\infty} B_n \sinh n\pi \sin ny$$ $\boxed{1\,pt}$

$$\Rightarrow B_n \sinh n\pi = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin ny \, dy$$ $\boxed{2\,pts}$

$$\Rightarrow B_n = \frac{2(1-(-1)^n)}{n\pi \sinh n\pi}.$$ $\boxed{2\,pts}$

c) [**5 points**] Use the solutions in parts a) and b) to find the solution of the problem:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x, y \in (0, \pi),$$

$$u(0, y) = 0, \quad u(\pi, y) = 1, \quad y \in (0, \pi),$$

$$u(x, 0) = 0, \quad u(x, \pi) = 1, \quad x \in (0, \pi).$$

Using the Superposition Principle, $\boxed{1\text{ pt}}$

$$u(x,y) = u_1(x,y) + u_2(x,y) \qquad \boxed{2\text{ pts}}$$

$$= \sum_{n=1}^{\infty} \left(A_n \sinh ny \sin nx + B_n \sinh nx \sin ny \right) \qquad \boxed{1\text{ pt}}$$

$$= \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n\pi \sinh n\pi} \left[\sinh ny \sin nx + \sinh nx \sin ny \right] \qquad \boxed{1\text{ pt}}$$

$\sim$

5. **[20 points]** Consider a vibrating circular membrane governed by the problem:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}, \qquad 0 < r < 1, \qquad t > 0,$$

$$u(1,t) = 0, \qquad t > 0,$$

$$u(r,0) = 0, \quad u_t(r,0) = 1, \qquad 0 < r < 1.$$

Use separation of variables to find an expression for $u(r,t)$.

Ans: Let $u(r,t) = R(r)\,T(t)$.

DE: $R''T + \frac{1}{r}R'T = RT''$

$\Rightarrow \dfrac{R'' + \frac{1}{r}R'}{R} = \dfrac{T''}{T} = -\lambda$ $\boxed{2\text{pts}}$

I) $R'' + \frac{1}{r}R' + \lambda R = 0$

$\Rightarrow r^2R'' + rR' + \lambda r^2 R = 0$

This is a Bessel DE with order $\nu = 0$.

$\Rightarrow \lambda = \alpha^2$ and

$R(r) = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$ $\boxed{2\text{pts}}$

Since $u(r,t)$ is bounded,

$c_2 = 0$. $\boxed{1\text{pt}}$

Also, $u(1,t) = 0$

$\Rightarrow R(1) = 0$

Hence, α satisfies $J_0(\alpha) = 0$:

$\boxed{\lambda_n = \alpha_n^2}$ satisfying $J_0(\alpha_n) = 0$. $\boxed{1\text{pt}}$

$\boxed{R_n(r) = J_0(\alpha_n r)}$, $n = 1, 2, \ldots$

II) $T'' + \lambda T = 0$; $\lambda = \alpha_n^2$

$\Rightarrow T'' + \alpha_n^2 T = 0$ $\boxed{2\text{pts}}$

$\Rightarrow T = c_1 \cos \alpha_n t + c_2 \sin \alpha_n t$

As $u(r,0) = 0 \Rightarrow T(0) = 0$, $\boxed{c_1 = 0}$

$\Rightarrow \boxed{T_n(t) = \sin \alpha_n t}$ $\boxed{2\text{pts}}$

Now, $u(r,t) = \sum_{n=1}^{\infty} A_n \sin \alpha_n t \, J_0(\alpha_n r)$ $\boxed{2\text{pts}}$

$\to$ Find A_n using $u_t(r,0) = 1$:

$u_t(r,t) = \sum_{n=1}^{\infty} A_n \alpha_n \cos \alpha_n t \, J_0(\alpha_n r)$

$u_t(r,0) = 1$ gives

$1 = \sum_{n=1}^{\infty} A_n \alpha_n J_0(\alpha_n r)$ $\boxed{2\text{pts}}$

$\Rightarrow \alpha_n A_n = \dfrac{2}{J_1^2(\alpha_n)} \displaystyle\int_0^1 x J_0(\alpha_n x)\,dx$

Set $t = \alpha_n x \Rightarrow dt = \alpha_n dx$.

$\Rightarrow A_n = \dfrac{2}{\alpha_n^3 J_1^2(\alpha_n)} \displaystyle\int_0^{\alpha_n} t J_0(t)\,dt$

$= \dfrac{2}{\alpha_n^3 J_1^2(\alpha_n)} \left(t J_1(t) \right)\Big|_0^{\alpha_n}$

$= \dfrac{2}{\alpha_n^2 J_1(\alpha_n)}$. $\boxed{4\text{pts}}$

6. **[20 points]** Find the temperature $u(r,\theta)$ in a sphere by solving the following problem:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0, \qquad 0 < r < 1, \qquad 0 < \theta < \pi,$$

$$u(1,\theta) = g(\theta), \qquad 0 < \theta < \pi.$$

Ans: Let $u(r,\theta) = R(r)\Theta(\theta)$.

DE: $R''\Theta + \frac{2}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + \frac{\cot\theta}{r^2}R\Theta' = 0$

$\Rightarrow \dfrac{R'' + \frac{2}{r}R'}{R} = -\dfrac{1}{r^2}\cdot\dfrac{\Theta'' + \cot\theta\,\Theta'}{\Theta}$

$\Rightarrow \dfrac{r^2R'' + 2rR'}{R} = -\dfrac{\Theta'' + \cot\theta\,\Theta'}{\Theta} = \lambda$ $\boxed{3\text{ pts}}$

I) $\Theta'' + \cot\theta\,\Theta' + \lambda\Theta = 0$

is a Legendre DE!

Put $x = \cos\theta$. It simplies to

$(1-x^2)\Theta''(x) - 2x\Theta'(x) + \lambda\Theta(x) = 0$ $\boxed{3\text{ pts}}$

Hence, $\boxed{\lambda_n = n(n+1)}$, $n = 0, 1, \dots$ $\boxed{2\text{ pts}}$

and $\Theta_n(x) = P_n(x)$.

$\Rightarrow \boxed{\Theta_n(\theta) = P_n(\cos\theta)}$ $\boxed{2\text{ pts}}$

II) $r^2R'' + 2rR' - \lambda R = 0$,

$\lambda_n = n(n+1)$.

$r^2R'' + 2rR' - n(n+1)R = 0$ $\boxed{2\text{ pts}}$

is a Cauchy-Euler DE,

AE: $m(m-1) + 2m - n(n+1) = 0$

$\Rightarrow m^2 + m - n(n+1) = 0$

$[m-n][m+(n+1)] = 0$.

$\Rightarrow m = n$ or $m = -(n+1)$.

$R(r) = c_1 r^n + c_2 r^{-(n+1)}$. $\boxed{2\text{ pts}}$

Since $u(r,\theta)$ is bounded,

$c_2 = 0$. $\boxed{2\text{ pts}}$

$\Rightarrow R_n(r) = r^n$

$\Rightarrow u(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$ $\boxed{2\text{ pt}}$

$\rightarrow$ Find A_n using $u(1,\theta) = g(\theta)$

$g(\theta) = \sum_{n=0}^{\infty} A_n P_n(\cos\theta)$

$\Rightarrow A_n = \dfrac{2n+1}{2}\displaystyle\int_0^{\pi} g(\theta)P_n(\cos\theta)\sin\theta$ $\boxed{2\text{ pts}}$

7. **[20 points]** Use the Laplace transform to solve the problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad x > 0, \quad t > 0$$

$$u(0,t) = 1, \quad \lim_{x\to\infty} u(x,t) = 0, \quad t > 0,$$

$$u(x,0) = e^{-x}, \quad u_t(x,0) = 0, \quad x > 0.$$

Ans: Let $\mathcal{L}\{u(x,t)\} = U(x,s)$.

DE: $\mathcal{L}\{u_{xx}\} = \mathcal{L}\{u_{tt}\}$ gives

$$\frac{d^2}{dx^2} U(x,s) = s^2 U(x,s) - s\, u(x,0) - u_t(x,0)$$

$$\Rightarrow \quad \frac{d^2}{dx^2} U - s^2 U = -s e^{-x}. \quad \boxed{3\text{pts}}$$

Now, $U = U_c + U_p$. $\boxed{1\text{pt}}$

I) U_c satisfies

$$\frac{d^2}{dx^2} U_c - s^2 U_c = 0$$

$$\Rightarrow \boxed{U_c = c_1 e^{sx} + c_2 e^{-sx}.} \quad \boxed{3\text{pts}}$$

II) $U_p = Ae^{-x}$ satisfies

$$\frac{d^2}{dx^2} U_p - s^2 U_p = -s e^{-x}$$

$$A e^{-x} - s^2 e^{-x} = -s e^{-x}$$

$$\Rightarrow \quad A = \frac{s}{s^2-1}.$$

Then,

$$U(x,s) = c_1 e^{sx} + c_2 e^{-sx} + \frac{s}{s^2-1} e^{-x}. \quad \boxed{1\text{pt}}$$

i) As $\lim_{x\to\infty} u(x,t) = 0$,

$\lim_{x\to\infty} U(x,s) = 0$ $\boxed{2\text{pts}}$

$\Rightarrow \boxed{c_1 = 0}$

ii) $u(0,t) = 1 \Rightarrow U(0,s) = \frac{1}{s}$ $\boxed{2}$

$$\Rightarrow \frac{1}{s} = c_2 + \frac{s}{s^2-1} \Rightarrow c_2 = \frac{1}{s} - \frac{s}{s^2-1}$$

$$\Rightarrow U(x,s) = \left(\frac{1}{s} - \frac{s}{s^2-1}\right) e^{-sx} + \frac{s}{s^2-1} e^{-x} \quad \boxed{2}$$

$$\Rightarrow u(x,t) = \left(1 - \cosh(t-x)\right) \mathcal{U}(t-x) + e^{-x}\cosh t.$$

$\boxed{3\text{pts}}$

8. a) [**10 points**] Find the Fourier transform of

$$f(x) = \begin{cases} 0, & x < -2 \\ -1, & -2 < x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$

Ans: $\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\, dx$ $\boxed{2pts}$

$$= \int_{-2}^{0} -1 \cdot e^{i\alpha x}\, dx + \int_{0}^{2} 1 \cdot e^{i\alpha x}\, dx \quad \boxed{2pts}$$

$$= \frac{-1}{i\alpha}\, e^{i\alpha x}\Big|_{-2}^{0} + \frac{1}{i\alpha}\, e^{i\alpha x}\Big|_{0}^{2} \quad \boxed{2pts}$$

$$= \frac{-2 + e^{2\alpha i} + e^{-2\alpha i}}{i\alpha} \quad \boxed{2pts}$$

$$= \frac{2(\cos 2\alpha - 1)}{i\alpha} \quad \boxed{2pts}$$

(b) [**8 points**] Use the Fourier transform to solve the initial value problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad -\infty < x < \infty, \ t > 0,$$

$$u(x,0) = g(x), \qquad -\infty < x < \infty.$$

Ans: Let $\mathcal{F}\{u(x,t)\} = U(\alpha,t)$.

DE: $\mathcal{F}\{u_{xx}\} = \mathcal{F}\{u_t\}$

$$-\alpha^2 U(\alpha,t) = \frac{d}{dt} U(\alpha,t) \qquad \boxed{3\text{pts}}$$

$$\Rightarrow \quad U(\alpha,t) = c e^{-\alpha^2 t}. \qquad \boxed{2\text{pts}}$$

As $u(x,0) = g(x)$, it follows that $U(\alpha,0) = G(\alpha)$.

Hence, $G(\alpha) = c e^{0} \Rightarrow c = G(\alpha)$.

Then, $U(\alpha,t) = G(\alpha) e^{-\alpha^2 t}$. $\boxed{1\text{pt}}$

Now, $u(x,t) = \mathcal{F}^{-1}\{G(\alpha) e^{-\alpha^2 t}\}$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\alpha) e^{-\alpha^2 t} e^{-i\alpha x} \, d\alpha$$

$\boxed{2\text{pts}}$

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☺ The exam is wonderful!

I liked it; I hope you liked it too ☺.

Formula Sheet for Math 301 Final Exam – 162

1. Two recurrence relations

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x), \qquad \frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

2. The Fourier Bessel series of f defined on the interval $(0,b)$ is $f(x) = \sum\limits_{i=1}^{\infty} c_i J_n(\alpha_i x)$, where $c_i = \dfrac{2}{b^2 J_{n+1}^2(\alpha_i b)} \int\limits_0^b x J_n(\alpha_i x) f(x)dx$ and α_i are defined by $\boxed{J_n(\alpha b) = 0}$.

3. The Fourier Bessel series of f defined on the interval $(0,b)$ is $f(x) = \sum\limits_{i=1}^{\infty} c_i J_n(\alpha_i x)$, where $c_i = \dfrac{2\alpha_i^2}{(\alpha_i^2 b^2 - n^2 + h^2) J_n^2(\alpha_i b)} \int\limits_0^b x J_n(\alpha_i x) f(x)dx$ and α_i are defined by $\boxed{hJ_n(\alpha b) + \alpha b J_n'(\alpha b) = 0}$.

4. The Fourier Bessel series of f defined on the interval $(0,b)$ is $f(x) = c_1 + \sum\limits_{i=2}^{\infty} c_i J_0(\alpha_i x)$, where $c_1 = \dfrac{2}{b^2} \int\limits_0^b x f(x)dx$, $c_i = \dfrac{2}{b^2 J_0^2(\alpha_i b)} \int\limits_0^b x J_0(\alpha_i x) f(x)dx$ and α_i are defined by $\boxed{J_0'(\alpha b) = 0}$.

5. The Fourier–Legendre series of f defined on the interval $(-1,1)$ is

$f(x) = \sum\limits_{n=0}^{\infty} c_n P_n(x)$, where $c_n = \frac{2n+1}{2} \int\limits_{-1}^{1} f(x) P_n(x)dx$. $\mathrm{P}_2(x) = \dfrac{1}{2}(3x^2 - 1)$

$\mathrm{P}_3(x) = \dfrac{1}{2}(5x^3 - 3x)$, $P_4(x) = \dfrac{1}{8}(35x^4 - 30x^2 + 3), P_5(x) = \dfrac{1}{8}(63x^5 - 70x^3 + 15x)$

6. Fourier transform: $\mathcal{F}\{f(x)\} = \int\limits_{-\infty}^{\infty} f(x) e^{i\alpha x} dx = F(\alpha)$

7. Inverse Fourier transform: $\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} F(\alpha) e^{-i\alpha x} d\alpha = f(x)$

8. Fourier sine transform: $\mathcal{F}_s\{f(x)\} = \int\limits_0^{\infty} f(x)\sin(\alpha x)dx = F(\alpha)$

9. Inverse Fourier sine transform: $\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi} \int\limits_0^{\infty} F(\alpha)\sin(\alpha x)d\alpha = f(x)$

10. Fourier cosine transform: $\mathcal{F}_c\{f(x)\} = \int\limits_0^{\infty} f(x)\cos(\alpha x)dx = F(\alpha)$

11. Inverse Fourier cosine transform: $\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi} \int\limits_0^{\infty} F(\alpha)\cos(\alpha x)d\alpha = f(x)$

12. $\mathcal{F}\{f''(x)\} = -\alpha^2 F(\alpha)$ AND $\mathcal{F}\{\dfrac{\partial^2 u(x,t)}{\partial x^2}\} = -\alpha^2 U(\alpha,t)$

13. $\mathcal{F}_s\{f''(x)\} = -\alpha^2 F(\alpha) + \alpha f(0)$ AND $\mathcal{F}_s\{\dfrac{\partial^2 u(x,t)}{\partial x^2}\} = -\alpha^2 U(\alpha,t) + \alpha u(0,t)$

14. $\mathcal{F}_c\{f''(x)\} = -\alpha^2 F(\alpha) - f'(0)$ AND $\mathcal{F}_c\{\dfrac{\partial^2 u(x,t)}{\partial x^2}\} = -\alpha^2 U(\alpha,t) - u_x(0,t)$

15. $\mathcal{L}\{f(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{f(t+a)\}$, $\mathcal{L}^{-1}\{F(s)e^{-as}\} = f(t-a)\mathcal{U}(t-a)$

16. $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$, $\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$