King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 301 - Exam I - Term 162

Duration: 120 minutes

Name: Solution

ID Number:

Section Number:

Serial Number:

Class Time:

Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 7 pages of problems (Total of 7 Problems)

Question Number	Points	Maximum Points
1		6
2		13
3		10
4		16
5		10
6		10
7		10
Total		75

1. **[6 points]**Find parametric equations of the tangent line of the vector function

$$\mathbf{r}(t) = t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}$$

at $t = \pi/3$.

Ans: • $r(\frac{\pi}{3}) = < \frac{\pi}{3}, \frac{1}{2}, \frac{\sqrt{3}}{2} >$ 1pt

$r'(t) = < 1, -\sin t, \cos t >$ 1pt

$r'(\frac{\pi}{3}) = < 1, -\frac{\sqrt{3}}{2}, \frac{1}{2} >$ 1pt

• Parametric Equations:

$x(t) = \frac{\pi}{3} + t$ 1pt

$y(t) = \frac{1}{2} - \frac{\sqrt{3}}{2} t$ 1pt

$z(t) = \frac{\sqrt{3}}{2} + \frac{1}{2} t.$ 1pt

2. Let $f(x, y) = \sin 2x \cos y$.

a)[**10 points**] Find the directional derivative of f at the point $(\pi/6, \pi/3)$ in the direction parallel to $3\,\mathbf{i} - 4\,\mathbf{j}$.

Ans: $\| 3i - 4j \| = 5$. Then, $u = \frac{1}{5}\langle 3, -4 \rangle$. 2pts

$\nabla f(x,y) = \langle 2\cos 2x \cos y, -\sin 2x \sin y \rangle$ 2pt

$\nabla f(\frac{\pi}{6}, \frac{\pi}{3}) = \langle \frac{1}{2}, -\frac{3}{4} \rangle$. 2pts

$D_u f(\frac{\pi}{6}, \frac{\pi}{3}) = \nabla f(\frac{\pi}{6}, \frac{\pi}{3}) \cdot u$ 2pts

$= \langle \frac{1}{2}, \frac{-3}{4} \rangle \cdot \frac{1}{5}\langle 3, -4 \rangle = \frac{9}{10}$ 2pts

b) [**3 points**] Find the direction in which the function f increases most rapidly at the point $(\pi/6, \pi/3)$.

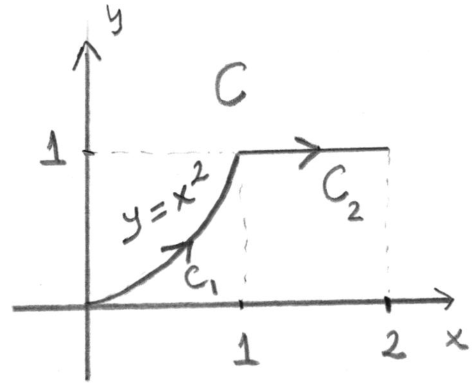
Ans: The vector is parallel to $v = \nabla f(\frac{\pi}{6}, \frac{\pi}{3})$:

$v = \langle \frac{1}{2}, \frac{-3}{4} \rangle$. 3pts

3. **[10 points]** Evaluate the line integral

$$\int_C (x^2 - y^2)\,dx + xy\,dy,$$

where C is the path given in the figure.



Ans:

$$\int_C (x^2-y^2)\,dx + xy\,dy = \int_{C_1} (x^2-y^2)\,dx + xy\,dy + \int_{C_2} (x^2-y^2)\,dx + xy\,dy$$

$C = C_1 \cup C_2$

- $$\int_{C_1} (x^2-y^2)\,dx + xy\,dy = \int_0^1 (x^2-x^4)\,dx + x x^2\, 2x\,dx \quad \underline{3\text{pts}}$$
  $$= \int_0^1 (x^2+x^4)\,dx = \frac{8}{15}. \quad \underline{1\text{pt}}$$

- $$\int_{C_2} (x^2-y^2)\,dx + xy\,dy = \int_1^2 (x^2-1)\,dx + x(1)(0) \quad \underline{3\text{pts}}$$
  $$= \frac{4}{3} \quad 1\text{pt}$$

Then,

$$\int_C (x^2-y^2)\,dx + xy\,dy = \frac{8}{15} + \frac{4}{3} = \frac{28}{15}.$$

$\underline{2\text{pts}}$

4. a)[**6 points**] Show that the line integral

$$\int_{(0,0,0)}^{(2,2,2)} xy^2\,dx + (x^2y + yz^2)\,dy + y^2z\,dz$$

is independent of path.

b) [**10 points**] Evaluate the integral in part a).

a) Ans: $P = xy^2$, $Q = x^2y + yz^2$, $R = y^2z$. We check

✓ $\partial_y P = 2xy = \partial_x Q$ 2pts

✓ $\partial_z P = 0 = \partial_x R$ 2pts or show that curl $\langle P, Q, R \rangle = \langle 0,0,0 \rangle$

✓ $\partial_z Q = 2yz = \partial_y R$ 2pts

Then, the integral is independent of path.

b) We find a potential function $\varphi = \varphi(x,y,z)$:

1) $\varphi_x = xy^2$, 2) $\varphi_y = x^2y + yz^2$, 3) $\varphi_z = y^2z$. 2pts

1) $\Rightarrow \varphi = \frac{1}{2}x^2y^2 + g(y,z)$

$\Rightarrow \varphi_y = x^2y + \partial_y g(y,z) = x^2y + yz^2$ from 2) 2pts

$\Rightarrow g(y,z) = \frac{1}{2}y^2z^2 + h(z)$

Hence, $\varphi = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2z^2 + h(z)$. 2pts

3) $\Rightarrow \varphi_z = y^2z + h'(z) = y^2z \Rightarrow h'(z) = 0$

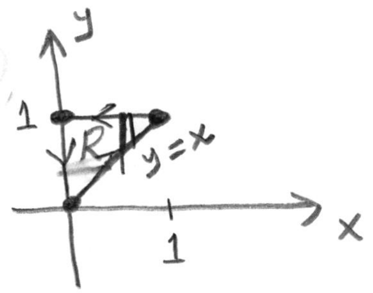
Take $h(z) = 0$. Then, $\varphi = \frac{1}{2}x^2y^2 + \frac{1}{2}y^2z^2$. 2pts

$$\int_{(0,0,0)}^{(2,2,2)} xy^2dx + (x^2y + yz^2)dy + y^2z\,dz = \varphi\Big|_{(0,0,0)}^{(2,2,2)} = 16.$$ 2pts

5. **[10 points]** Use Green's theorem to evaluate the closed line integral

$$\oint_C e^{x^2}\,dx + \ln(1+x^2)\,dy,$$

where C is the triangle with vertices $(0,0), (1,1), (0,1)$.



Ans: $P = e^{x^2}$, $Q = \ln(1+x^2)$

$$\oint_C e^{x^2}dx + \ln(1+x^2)\,dy$$

$$= \iint_R (\partial_x Q - \partial_y P)\,dA = \int_0^1\int_x^1 \frac{2x}{x^2+1}\,dy\,dx \quad \underline{\underline{2pts}} + \underline{\underline{2pts}}$$

$$= \int_0^1 \left.\frac{2xy}{x^2+1}\right|_x^1 dx$$

$$= \int_0^1 \left(\frac{2x}{x^2+1} - \frac{2x^2}{x^2+1}\right) dx \quad \underline{\underline{2pts}}$$

$$= \int_0^1 \left(\frac{2x}{x^2+1} - 2 - \frac{2}{x^2+1}\right) dx$$

$$= \ln(x^2+1) - 2x + 2\tan^{-1}x \Big|_0^1 \quad \underline{\underline{2pts}}$$

$$= \ln 2 - 2 + \frac{\pi}{2}. \quad \underline{\underline{2pts}}$$

6. **[10 points]** Let $\mathbf{F}(x, y, z) = x\,\mathbf{i} + xy^2\,\mathbf{j} + z\,\mathbf{k}$ and S be the portion of the plane $z = -2x + 6$ within the cylinder $x^2 + y^2 = 4$. Find the flux of $\mathbf{F}$ through S.

Ans:

$$\text{Flux} = \int_S F \cdot n \, dS \qquad \underline{2pts}$$

- We find n.: Let $g(x,y,z) = z + 2x - 6$.

Then, $$n = \frac{\nabla g}{\|\nabla g\|} = \frac{\langle 2, 0, 1\rangle}{\sqrt{5}} \qquad \underline{2pts}$$

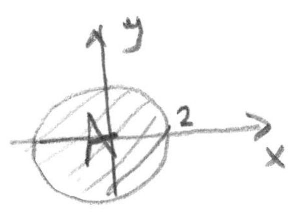
- The surface S: $z = f(x,y) = -2x + 6$

$$dS = \sqrt{1 + f_x^2 + f_y^2}\, dA = \sqrt{5}\, dA \qquad \underline{2pts}$$

- $$\text{Flux} = \int_A \langle x, xy^2, \underbrace{-2x+6}_{z}\rangle \cdot \frac{\langle 2, 0, 1\rangle}{\sqrt{5}} \sqrt{5}\, dA$$

$$= \int_A 6\, dA \qquad \underline{2pts}$$

$$= 6\pi(2)^2 = 24\pi. \qquad \underline{2pts}$$



7. **[10 points]** Use the divergence theorem to evaluate the flux

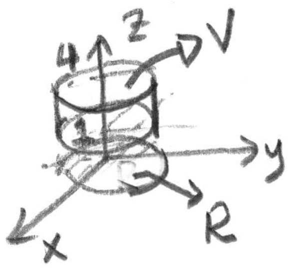
$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

where $\mathbf{F}(x,y,z) = y^2x\,\mathbf{i} + 4xe^z\,\mathbf{j} + x^2z\,\mathbf{k}$ and S is the boundary of the region D bounded by the cylinder $x^2+y^2=1$ and the planes $z=1$ and $z=4$.

Ans:

- $\operatorname{div} F = \nabla \cdot \langle y^2x, 4xe^z, x^2z\rangle = y^2 + x^2$ 2pts

- $\iint_S F\cdot n\, dS = \iiint_V \operatorname{div} F\, dV$ 2pts



$= \int_1^4 \iint_R (x^2+y^2)\,dx\,dy\,dz$ 1pt

$= 3\iint_R (x^2+y^2)\,dx\,dy$ 2pts

$= 3\int_0^{2\pi}\int_0^1 r^2 \cdot r\,dr\,d\theta$ 2pts

$= 3(2\pi)\cdot \frac{1}{4} r^4 \Big|_0^1$

$= \frac{3\pi}{2}.$ 1pt