King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics
MATH 280-01(Term 162)

**Final Exam**

May 26, 2017

NAME: ........................................................................

ID #: . . . . . . . . . . . . .

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| Question | Points | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 13 | |
| 3 | 13 | |
| 4 | 10 | |
| 5 | 13 | |
| 6 | 13 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 13 | |
| 11 | 10 | |
| 12 | 15 | |
| Total | 140 | |

Q1. Let $A$ be an $n \times n$ matrix. Fill in the following blanks:

a) If $A$ is nonsingular and $\lambda$ is an eigenvalue, then ...... is an eigenvalue of $A^{-1}$.

b) If $\lambda$ is an eigenvalue of $A$, then ...... is an eigenvalue of $A^T$.

c) If $\lambda$ is a complex eigenvalue of $A$, then ...... is an eigenvalue of $A$.

d) If $A$ is symmetric and $\lambda$ is an eigenvalue, the $\lambda$ is ......

e) If $A$ is symmetric positive definite and $\lambda$ is an eigenvalue, the $\lambda$ is ......

f) If $A$ is singular, then $\lambda =$ ...... is an eigenvalue of $A$.

g) If $A$ has $n$ distinct eigenvalues, then $A$ is ......

h) The sum of all the eigenvalues of $A$ is equal to ......

i) The product of all the eigenvalues of $A$ is equal to ......

j) If $B$ is similar to $A$ and $\lambda$ is an eigenvalue of $A$, then ...... is an eigenvalue of $B$.

Q2. Diagonalize the following matrix:

$$A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix}.$$

Q3. Show that if $A$ ia any $n \times n$ nonsingular matrix, then $A^T A$ is **symmetric positive definite**

Q4. Given the quadratic polynomial

$$f(x, y, z) = 2x^2 - 2xy - 4xz + y^2 + 2yz + 3z^2 - 2x + 2z$$

Write $f(x, y, z)$ in the form

$$f(x, y, z) = X^T AX + \mathbf{b}^T X$$

where $X = (x, y, z)^T$, $A$ is a real symmetric matrix, and $\mathbf{b}$ is some constant vector.

Q5. Let $\mathbf{u}$ be a **unit** vector in $\mathbb{R}^n$ and let $H = I - 2\mathbf{u}\mathbf{u}^T$. Show that $H$ is both **orthogonal** and **symmetric**.

Q6. (a) Verify that the following defines an inner product on $\mathbb{R}^3$:

$$\langle x, y \rangle = 3x_1y_1 + 2x_2y_2 + x_3y_3$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

(b) Consider the vectors

$$u = \begin{bmatrix} 1 \\ a \\ -1 \end{bmatrix}, \; v = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}.$$

Find $a$ such that the two vectors are orthogonal.

Q7. Given $A$ and $B$ in $\mathbb{R}^{2\times 2}$ define

$$\langle A, B \rangle = tr(AB)$$

where $tr$ is the trace of a matrix. Is this a valid inner product (explain)?.

Q8. Suppose $A$ is an $m \times n$ matrix whose columns are linearly independent. What is the nullity of $A$ (explain)?

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Q9. For each $f \in C[0,1]$ define

$$T(f) = \begin{bmatrix} f(0) \\ f(1)+1 \end{bmatrix}$$

Is $T(f)$ a linear transformation from $C[0,1]$ into $\mathbb{R}^2$?

Q10. Consider the function $F : \mathbb{P}_3 \longrightarrow \mathbb{R}^3$ that maps a polynomial in $\mathbb{P}_3$ to the vector

$$\begin{bmatrix} p(3) \\ p'(3) \\ p''(3) \end{bmatrix}$$

(a) Show that $F$ is a linear transformation.

(b) Find the standard matrix representation of $F$.

Q11. Determine whether the following vectors are linearly independent in $\mathbb{P}_3$:

$$x + 2,\ x + 1,\ x^2 - 1$$

Q12. Given the basis

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

for $\mathbb{R}^3$, use the Gram-Schmidt process to obtain an orthonormal basis.