KFUPM/ Department of Mathematics & Statistics/ 162/ MATH 232/ Test

Name: **ID #:** **Serial #:**

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1. Give an example of 3 subsets A, B, C of $\mathbb{N}$ such that $|A \cap B| = 0$, $|B \cap C| = 1$, $|C \cap A| = 2$.

Solution. Take for example $A = \{1, 2\}$, $B = \{3\}$, $C = \{1, 2, 3\}$.

2. For statements P and Q, which of the following is a tautology? Justify.

a $(P \vee Q) \Rightarrow (P \wedge Q)$

b $P \Rightarrow (P \Rightarrow Q)$

Solution. We have (at least) 3 ways of solving this.

For convenience, let A be the statement $(P \vee Q) \Rightarrow (P \wedge Q)$ and B be the statement $P \Rightarrow (P \Rightarrow Q)$.

(i) Using a Truth Table:

P	Q	$P \vee Q$	$P \wedge Q$	A	$P \Rightarrow Q$	B
T	T	T	T	T	T	T
T	F	T	F	F	F	F
F	T	T	F	F	T	T
F	F	F	F	T	T	T

Since neither A nor B is always True, we infer that none of them is a Tautology.

(ii) Using properties of Logical Equivalence:

a

$$\begin{aligned}
A &\equiv \sim (P \vee Q) \vee (P \wedge Q) \equiv ((\sim P) \wedge (\sim Q)) \vee (P \wedge Q) \\
&\equiv ((\sim P) \vee (P \wedge Q)) \wedge ((\sim Q) \vee (P \wedge Q)) \\
&\equiv ((\sim P) \vee P) \wedge ((\sim P) \vee Q) \wedge ((\sim Q) \vee P) \wedge ((\sim Q) \vee Q) \\
&\equiv ((\sim P) \vee Q) \wedge ((\sim Q) \vee P) \ \text{ (because } (\sim P) \vee P \text{ and } ((\sim Q) \vee Q) \text{ are always True)} \\
&\equiv P \Leftrightarrow Q, \text{ and this is not a Tautology (e.g. take } P \text{ True and } Q \text{ False).}
\end{aligned}$$

b

$$\begin{aligned}
B &\equiv (\sim P) \vee ((\sim P) \vee Q) \\
&\equiv ((\sim P) \vee (\sim P)) \vee Q \equiv (\sim P) \vee Q \\
&\equiv P \Rightarrow Q \text{ which is not a Tautology (e.g. take } P \text{ True and } Q \text{ False).}
\end{aligned}$$

(iii) By inspection:

a If we take P True and Q False, then $P \vee Q$ is True and $P \wedge Q$ is False, hence A is False and therefore it is not a Tautology.

b If we take P True and Q False, then $P \Rightarrow Q$ is False, hence B is False and therefore it is not a Tautology.

3. Let $S = \{3, 5\}$. Consider the quantified statement: For every $x \in S$ and $y \in S$, $xy - 2$ is prime.

a Is this quantified statement True or False? Justify.

b Write down (in symbols or in words) the negation of the quantified statement.

Solution.

a If $(x, y) = (3, 3)$, then $xy - 2 = 7$; if $(x, y) = (3, 5)$ (or $(5, 3)$), then $xy - 2 = 13$; and if $(x, y) = (5, 5)$, then $xy - 2 = 23$. Since $7, 13, 23$ are prime, we infer that the quantified statement is True.

b In words: There exist $x \in S$ and $y \in S$ such that $xy - 2$ is not prime. (Or, "$xy - 2$ is not prime for some x and y in S.")

[In symbols: $\exists (x, y) \in S \times S$, such that $xy - 2$ is not prime.]

4. Let x and y be integers. Prove that if xy and $x + y$ are even, then both x and y are even.

- **Direct proof**: Suppose xy and $x + y$ are even, then there exist integers m, n such that $xy = 2m$ and $x + y = 2n$. This means $(x + 1)(y + 1) = xy + x + y + 1 = 2m + 2n + 1 = 2(m + n) + 1$, which is odd (since $m + n \in \mathbb{Z}$). Hence $x + 1$ and $y + 1$ are both odd, i.e. x and y are even. □

- **Proof by contrapositive**: Assume not both of x and y are even, i.e. assume that x or y is odd. W.L.O.G., we can assume that x is odd, i.e. $x = 2m + 1$ for some integer m. We have 2 cases:

  Case 1. y is even. Let $y = 2n$ (some n in $\mathbb{Z}$). Then $x + y = (2m + 1) + 2n = 2(m + n) + 1$, which is odd (since $m + n \in \mathbb{Z}$). Hence not both xy and $x + y$ are even.

  Case 2. y is odd. Let $y = 2n + 1$ ($n \in \mathbb{Z}$). Then $xy = (2m + 1)(2n + 1) = 2(2mn + m + n) + 1$, which is odd (since $2mn + m + n \in \mathbb{Z}$). Hence not both xy and $x + y$ are even.

- **Another proof by contrapositive (without cases)**: Assume not both of x and y are even, i.e. assume that x or y is odd. W.L.O.G., we can assume that x is odd, i.e. $x = 2m + 1$ for some integer m. Then $xy + (x + y) = (x + 1)y + x = (2m + 2)y + 2m + 1 = 2(my + m + y) + 1$, which is odd (since $my + m + y \in \mathbb{Z}$). Since the sum of xy and $(x + y)$ is odd, they cannot have the same parity, and therefore one of them must be odd. □