KFUPM/ Department of Mathematics & Statistics/ 162/ MATH 232/ Quiz 1

Name: **ID #:** **Serial #:**

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1. Prove that there are no odd integers a, b such that $4 \mid (3a^2 - b^2)$.

Solution. Suppose on the contrary that there are odd integers $a = 2h + 1$, $b = 2k + 1$ such that $4 \mid (3a^2 - b^2)$.
We have

$$\begin{aligned} 3a^2 - b^2 &= 4a^2 - \left(a^2 + b^2\right) \\ &= 4a^2 - \left(4h^2 + 4h + 1 + 4k^2 + 4k + 1\right) \\ &= 4\left(a^2 - h^2 - h - k^2 - k\right) - 2 \end{aligned}$$

Since $4 \mid (3a^2 - b^2)$ and $4 \mid 4(a^2 - h^2 - h - k^2 - k)$ ($\because a^2 - h^2 - h - k^2 - k \in \mathbb{Z}$) we obtain $4 \mid 2$, which is impossible. This contradiction shows that that there are no odd integers a, b such that $4 \mid (3a^2 - b^2)$. ■

2. Prove by induction that $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$ for every positive integer n.

Solution. Let $P(n)$ be the statement to be proved.
$P(1)$ is clearly true ($\because 1^3 = \left(\frac{1 \times (1+1)}{2}\right)^2$).
Assume that $P(k)$ is true for a positive integer k.
Then $1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left(\frac{k^2}{4} + k + 1\right) = \left(\frac{(k+1)(k+2)}{2}\right)^2$,
hence $P(k+1)$ is true.
By the induction principle, $P(n)$ is true for every positive integer n. ■