King Fahd University of Petroleum & Minerals

Department of Mathematics & Statistics

2016-2017 (Term 162)

Introduction to Sets and Structures (MATH 232)

Final Exam

Student Name: **ID** #: **Serial** #:

Question	Marks	Out of
1		15
2		15
3		15
4		15
5		15
6		15
7		15
8		15
Total		120

1. (a) Prove that the function $f : \mathbb{R} - \{2\} \longrightarrow \mathbb{R} - \{1\}$ defined by $f(x) = \dfrac{2x-1}{2x-4}$ is a bijection. What is the inverse function of f?

(b) Let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be functions such that $g \circ f = i_A$. Prove that f is one-to-one and that g is onto.

2. (a) Let A be a denumerable set. Prove that $|A| = |A \times A|$.

(b) State Schröder-Bernstein Theorem and use it to prove that if two nonempty sets A and B are such that $|A| < |B|$, then there does not exist any one-to-one function from B to A.

3. (a) Which of the following sets are denumerable: $\mathbb{Z}_2 \times \mathbb{Z}_3$, $\mathbb{Q} \times \mathbb{Q}$, $\mathbb{C}$? Justify.

(b) Prove that the set $\mathbb{R} \times \mathbb{R}$ is uncountable.

4. (a) Let d be a nonzero integer that divides the integers m and n. Prove that d divides every linear combination of a and b.

(b) Show that if m is an integer, then $\gcd(2m+1, 3m+1) = 1$.

(c) Two integers m and a are such that $m \geq 2$, $15a + 1 \equiv 0 \pmod{m}$ and $5a - 2 \equiv 0 \pmod{m}$. Find m.

5. (a) Find the smallest prime divisor of 793.

(b) Let n be an integer greater than 1 with prime power factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$. If n is a perfect cube, what can we say about the exponents $a_1, \ldots, a_r$? Justify.

6. (a) Let G be a group and let $n \geq 1$. Prove by induction that for all elements $a_1, \ldots, a_n$ in G, we have $(a_1 a_2 \cdots a_n)^{-1} = a_n^{-1} \cdots a_2^{-1} a_1^{-1}$.

(b) In the group $\left(\mathbb{Z}_p^*, \cdot\right)$, the inverse of $[p-1]$ is of the form $[p-k]$ for some positive integer k. Find k.

7. (a) Let G be a (finite) group containing subgroups H and K such that $|H| = 3$ and $|K| = 4$. What is the smallest possible order of G? Justify.

(b) Let G be an abelian group.

(i) Prove that $(ab)^3 = a^3b^3$ for all a and b in G.

(ii) Is the subset $H = \{a \in G : a^3 = e \text{ for some } a \in G\}$ a subgroup of G? Justify.

8. (a) Let R be a relation defined on $\mathbb{N}$ by aRb iff $5a - 2b \equiv 0 \pmod 3$. Is R an equivalence relation? Justify.

(b) Let R be a relation defined on $\mathbb{R} \times \mathbb{R}$ by $(a, b)\, R\, (c, d)$ iff $a \leq c$ and $b \leq d$, where $\leq$ is the usual order relation on the real numbers.

Prove that $(\mathbb{R} \times \mathbb{R}, R)$ is a poset. Is $(\mathbb{R} \times \mathbb{R}, R)$ well-ordered? Justify.