KFUPM/ Department of Mathematics & Statistics/ 162/ MATH 232/ Exam 2

Name: **ID #:** **Serial #:**

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1. [6pts] Prove that there are no positive integers a, n such that $a^2 + 2 = 2^n$.

Proof. Suppose to the contrary that there are positive integers a, n such that $a^2 + 2 = 2^n$. Since $n \geq 1$, we obtain that 2^n is even, and hence $2^n - 2$, i.e. a^2, is even. This means a is even. Put $a = 2k$ where k is a positive integer.
We have $4k^2 = 2^n - 2$ which gives $2k^2 + 1 = 2^n$. In this last equation, the LHS is odd but the RHS is even (as $n \geq 1$), and this is impossible. ■

2. [6pts] Show that the following statements are **false**.
(i) If a and b are positive irrational numbers, then $a + b$ is irrational.
(ii) If a and b are positive irrational numbers, then $a^2 + b$ is irrational.

[*You may use the fact that* $\sqrt{2}$ *is irrational.*]

Solution. Let $a = 2 - \sqrt{2}$, $b = \sqrt{2}$. Then b is irrational and also a is irrational (because if a were rational then so too would be $2 - a$, and this would mean $\sqrt{2}$ is rational). Clearly a and b are positive. We have:
(i) $a + b$ is rational ($a + b = 2$), which shows that the statement (i) is false.

(ii) $4b$ is irrational (otherwise b would be rational) and $a^2 + 4b$ is rational since $a^2 + 4b = \left(2 - \sqrt{2}\right)^2 + 4\sqrt{2} = 6$. This shows that (ii) is false.

3. [8pts] Prove by induction that the following statement is true for every integer $n \geq 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Proof. Let $P(n)$ be the statement above.

Base step: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2} \Leftrightarrow \frac{\sqrt{2}+1}{\sqrt{2}} > \sqrt{2} \Leftrightarrow \frac{\sqrt{2}-1}{\sqrt{2}} > 0$, which is true. Hence $P(2)$ is true.

Inductive step: Suppose $P(k)$ is true for some integer $k \geq 2$. We prove that $P(k+1)$ is true, i.e. that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$.

We have $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k+1}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$.
If we prove that $\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k+1}$, then we are done.
We have $\left(\sqrt{k} + \frac{1}{\sqrt{k+1}}\right)\sqrt{k+1} = \sqrt{k^2+k} + 1 \geq k + 1$ (because $k^2 + k \geq k^2$).
Hence $\sqrt{k} + \frac{1}{\sqrt{k+1}} \geq \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}$, as required. ■

4. [8pts] Let $F_1, F_2, \ldots$ be the sequence of integers defined recursively by:

$$F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2} \ (n \geq 3)$$

Prove by induction the following statement.

$$F_1 + F_2 + \cdots + F_n = F_{n+2} - 1 \text{ for all } n \in \mathbb{N}.$$

Proof. Let $P(n)$ be the above statement.

Base step: We have $F_3 = F_1 + F_2 = 2$, hence $F_1 = 1 = F_3 - 1$ and so $P(1)$ is true.

Inductive step: Assume that $P(k)$ is true for some $k \in \mathbb{N}$. We prove that $P(k+1)$ is true, i.e. that $F_1 + F_2 + \cdots + F_{k+1} = F_{k+3} - 1$.

We have $F_1 + F_2 + \cdots + F_{k+1} = F_1 + F_2 + \cdots + F_k + F_{k+1} = F_{k+2} + F_{k+1} - 1 = F_{k+3} - 1$ (using the recursive definition of the sequence $F_1, F_2, \ldots$), as required. ■

5. [6pts] Let R be a relation on a set A. Which of the following statements is TRUE? Either prove or give a counterexample to justify your answer.

(i) $R = (R^{-1})^{-1}$.
(ii) $R \cup R^{-1}$ is reflexive.
(iii) If R is transitive, then R^{-1} is also transitive.

Solution. (i) True: Let $a, b \in A$. Then $(a,b) \in R \Leftrightarrow (b,a) \in R^{-1} \Leftrightarrow (a,b) \in (R^{-1})^{-1}$. This proves that $R \subseteq (R^{-1})^{-1}$ and $(R^{-1})^{-1} \subseteq R$, i.e. that $R = (R^{-1})^{-1}$.

(ii) False: Let $A = \{0,1\}$ (as a subset of $\mathbb{Z}$) and $R = \{(0,1)\}$. Then $R \cup R^{-1} = \{(0,1), (1,0)\}$. Since $(0,0) \notin R \cup R^{-1}$, the relation $R \cup R^{-1}$ is not reflexive.
[*We could more simply just take R to be the empty relation $\emptyset$, then $R \cup R^{-1} = \emptyset$ which is not reflexive.*]

(iii) True: Let $a, b, c \in A$ be such that $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1}$. Then (b,a) and (c,b) are in R, and as R is transitive, (c,a) is also in R. Hence $(a,c) \in R^{-1}$, as required.

6. [6pts] Let S be the relation on $\mathbb{R}$ defined by xSy iff $xy \geq 0$.
(i) Is S reflexive?
(ii) Is S symmetric?
(iii) Is S transitive?
Justify your answers.

Solution. (i) Let x be a real number. Then $x^2 \geq 0$, hence xSx and therefore S is reflexive.

(ii) Let x, y be real numbers such that xSy. Then $xy \geq 0$ and hence $yx \geq 0$ (since $xy = yx$). Therefore ySx, i.e. S is symmetric.

(iii) Let $x = 1$, $y = 0$, $z = -1$. Then xSy and ySz (because $xy = yz = 0$), but $xz = -1 \ngeq 0$, i.e $(x,z) \notin S$. Hence S is not transitive.