KFUPM/ Department of Mathematics & Statistics/ 162/ MATH 232/ Exam 1

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1. [6pts] *Let* $A = \{1, 2, 3, 4\}$. *Give an example of two sets* S *and* B *such that:* $S \subseteq P(A)$, $|S| = 3$, $B \in S$, *and* $|B| = 2$.

Solution. We can take $B = \{1, 2\}$, $S = \{\emptyset, \{1\}, B\}$.

2. [6pts] *Let* P, Q, R *be statements. Show that the statements* $(P \vee Q) \Rightarrow (\sim R)$ *and* $R \Rightarrow ((\sim P) \wedge (\sim Q))$ *are logically equivalent.*

Proof. We can use a Truth Table. We can also use properties of logical equivalence, as follows.

$$(P \vee Q) \Rightarrow (\sim R) \quad \equiv \quad (\sim (P \vee Q)) \vee (\sim R)$$

$$R \Rightarrow ((\sim P) \wedge (\sim Q)) \quad \equiv \quad (\sim R) \vee ((\sim P) \wedge (\sim Q))$$
$$(\sim R) \vee (\sim (P \vee Q)) \text{ (by De Morgan's law)}$$

Since $(\sim (P \vee Q)) \vee (\sim R) \equiv (\sim R) \vee (\sim (P \vee Q))$, we obtain that $(P \vee Q) \Rightarrow (\sim R)$ and $R \Rightarrow ((\sim P) \wedge (\sim Q))$ are logically equivalent.

3. [8pts] (i) *Let* a *and* b *be real numbers. Prove that either* $\frac{a+b}{2} \geq a$ *or* $\frac{a+b}{2} > b$.

Proof. We have:
Either $a \leq b$, and then $a + b \geq 2a$, so that $\frac{a+b}{2} \geq a$;
or $a > b$, and then $a + b > 2b$, so that $\frac{a+b}{2} > b$.

(ii) *Let* a *and* b *be positive real numbers. Prove that if* $a \neq b$, *then* $\frac{a}{b} + \frac{b}{a} > 2$.

Proof. Suppose $a \neq b$. We have $\frac{a}{b} + \frac{b}{a} - 2 = \frac{a^2 - 2ab + b^2}{ab} = \frac{(a-b)^2}{ab}$. Since $a \neq b$ we obtain that $(a-b)^2 > 0$, and since a and b are both positive, we obtain that ab is positive. Hence $\frac{(a-b)^2}{ab} > 0$, i.e. $\frac{a}{b} + \frac{b}{a} - 2 > 0$. This shows that $\frac{a}{b} + \frac{b}{a} > 2$.

4. [6pts] *Prove by cases that if* x *is a real number such that* $x^2 - x - 6 < 0$, *then* $-2 < x < 3$.

Proof. We have $x^2 - x - 6 = (x + 2)(x - 3)$.
Suppose $x^2 - x - 6 < 0$. Then $(x + 2)$ and $(x - 3)$ have opposite signs.
Case $x + 2 > 0$ and $x - 3 < 0$. In this case $-2 < x < 3$.
Case $x + 2 < 0$ and $x - 3 > 0$. In this case $x < -2$ and $x > 3$, which is impossible.
The only possibility therefore is that $-2 < x < 3$.

5. [6pts] (i) *A perfect square is divided by 5. What are all the possible remainders?*

Solution. Let x be an arbitrary integer. Then x is congruent to 0 or ± 1 or ± 2 modulo 5, and hence x^2 is congruent to 0 or 1 or 4 modulo 5.
The possible remainders are therefore 0 or 1.

(ii) *Prove that if* $a \in Z$, *then* $5 \nmid (a^2 + 2)$.

Proof. Assume to the contrary that $5 \mid (a^2 + 2)$ for some integer a.
Then $a^2 + 2 \equiv 0 \pmod 5$, and hence $a^2 \equiv 3 \pmod 5$, which contradicts the fact we obtained in Part (i) that the only possible remainders when a perfect square is divided by 5 are 0 or 1 or 4.
We conclude that $5 \nmid (a^2 + 2)$.

6. [8pts] *Let* A, B, *and* C *be sets.*

(i) *Prove that* $A \cup B \subseteq A \cap B$ *if and only if* $A = B$.

Solution. Suppose that $A \cup B \subseteq A \cap B$. Then $A \subseteq A \cup B \subseteq A \cap B \subseteq B$.
Similarly, $B \subseteq A$. Hence $A = B$.
For the converse, it is clear that if $A = B$, then $A \cup B = A = A \cap B$, and so $A \cup B \subseteq A \cap B$.

(ii) *Prove that if* $A = B$ *then* $A \times C = B \times C$. *Is the converse true? Justify.*

Proof. Suppose $A = B$.
To prove $A \times C = B \times C$, we need only prove that $A \times C \subseteq B \times C$ (the proof of the reverse inclusion $B \times C \subseteq A \times C$ is similar).
Let $(x, y) \in A \times C$, then $x \in A$ and $y \in C$. Since $A = B$, we obtain $x \in B$. Hence $(x, y) \in B \times C$, as required.

The converse is: If $A \times C = B \times C$ then $A = B$. This is false:
Take for example $A = C = \emptyset$ and $B = \mathbb{N}$. Then $A \times C = B \times C = \emptyset$, but $A \neq B$.