KFUPM, DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 202 : TEST 2, T 162, MARCH 07, 2017

Name : ....

ID : .... Solution

Exercise 1. Solve the Bernoulli DE :

$$y' = y(1 + ye^x).$$

Solution.

The DE is equivalent to $y' - y = y^2 e^x$; that is

$y^{-2}y' - y^{-1} = e^x$ (for $y \neq 0$).

We use the substitution $v = y^{-1}$; then $\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$.

Hence the DE is transformed into:

$$-\frac{dv}{dx} - v = e^x \iff \frac{dv}{dx} + v = -e^x;$$ which is

a linear DE.

- $\mu(x) = e^{\int dx} = e^x$ is an integrating factor of the above DE. Thus, multiplying by e^x both sides of the DE, we get $\frac{d}{dx}[e^x v] = -e^{2x}$; this leads to $e^x v = -\frac{1}{2}e^{2x} + c$

$$\longrightarrow v = -\frac{1}{2}e^x + \frac{c}{e^x}.$$

It follows that $\boxed{y = v^{-1} = \frac{1}{-(1/2)e^x + c\,e^{-x}}}$

$y = 0$ is a singular sol.

Exercise 2. Solve the DE :

$$y' = -\frac{\sin y - y\sin x}{x\cos y + \cos x}.$$

Solution. The differential form of the DE is:

$$(\sin y - y\sin x)\,dx + (x\cos y + \cos x)\,dy = 0.$$

Letting $M(x,y) = \sin y - y\sin x$ and $N(x,y) = x\cos y + \cos x$,

we get $M_y = \cos y - \sin x$ and $N_x = \cos y - \sin x$.

As $M_y = N_x$, the given DE is exact.

Let us find $\varphi(x,y)$ such that $\begin{cases} \dfrac{\partial\varphi}{\partial x} = \sin y - y\sin x & (1) \\ \dfrac{\partial\varphi}{\partial y} = x\cos y + \cos x & (2) \end{cases}$

$(1) \longrightarrow \varphi(x,y) = x\sin y + y\cos x + h(y)$.

So, substituting $\varphi(x,y)$ by its value in (2), we obtain:

$$x\cos y + \cos x + h'(y) = x\cos y + \cos x; \text{ Thus } h'(y) = 0$$

It follows that $h(y) = c$.

Consequently $\varphi(x,y) = x\sin(y) + y\cos(x)$ is a potential associated with the given exact DE.

Conclusion: The solutions of the DE are given implicitely by the relation $\boxed{x\sin(y) + y\cos x = c}$, where c is a constant.

Exercise 3. Solve the following IVP :

$$(y\mathrm{d}x + x(\ln x - \ln y - 1)\mathrm{d}y = 0, \quad y(1) = e)$$

Solution.

The standard form of the DE is:

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)\frac{1}{\ln x - \ln(y) - 1} = +\left(\frac{y}{x}\right)\frac{1}{1+\ln(y)-\ln(x)}$$

$$= \left(\frac{y}{x}\right)\frac{1}{1+\ln\left(\frac{y}{x}\right)}$$

So the DE is homogeneous.

We use the substitution $v = \frac{y}{x}$ (i.e., $y = xv$).

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So the DE is transformed into:

$$v + x\frac{dv}{dx} = v \cdot \frac{1}{1+\ln(v)}$$

$$\Longleftrightarrow x\frac{dv}{dx} = \frac{v}{1+\ln(v)} - v = \frac{-v\ln(v)}{1+\ln(v)}$$

$$\Longleftrightarrow -\left(\frac{1+\ln(v)}{v\ln(v)}\right)dv = \left(\frac{1}{x}\right)dx$$

By integrating, we get $\int\left(\frac{1}{v}\ln(v) + \frac{1}{v}\right)dv = -\ln(x) + c$ one may suppose that (here we are on an interval centred at 1, it is contained in $(0,2)$).

Hence $\frac{1}{2}(\ln(v))^2 + \ln(v) = -\ln(x) + C$

Solution of Ex 3

The standard form of the DE is

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)\frac{1}{\ln(x)-\ln(y)-1} = \left(\frac{y}{x}\right)\frac{1}{1+\ln(y)-\ln(x)}$$

$$= \left(\frac{y}{x}\right)\frac{1}{1+\ln\left(\frac{y}{x}\right)}$$

So the DE is homogeneous.

We use the substitution $v = \frac{y}{x}$ (i.e., $y = xv$):

$\frac{dy}{dx} = v + x\frac{dv}{dx}$. So the DE is transformed into

$$v + x\frac{dv}{dx} = v\left(\frac{1}{1+\ln v}\right) \Leftrightarrow x\frac{dv}{dx} = \frac{v}{1+\ln(v)} - v$$

$$\Leftrightarrow x\frac{dv}{dx} = \frac{-v\ln(v)}{1+\ln(v)}$$

$$\Leftrightarrow \left(\frac{1+\ln(v)}{v\ln(v)}\right)dv = -\frac{1}{x}dx$$

By integrating, we get $\int \frac{1}{v}\left(\frac{1}{\ln(v)}\right)dv + \int\frac{1}{v}dv = -\ln(x) + c_1$

(here we are on an interval centred at 1, one may suppose that it is contained in $(0,+\infty)$).

- $\int \frac{1}{v}(\ln(v))^{-1}dv = \int \frac{(1/v)}{\ln(v)}dv = \ln(\ln(v))$
- $\int\frac{1}{v}dv = \ln(v)$.

So the integration yields: $\ln(\ln(v)) + \ln(v) = \ln\left(\frac{1}{x}\right) + c_1$

$$\rightarrow \ln(\ln(v) + v) = \ln(1/x) + c_1$$

$$\rightarrow \boxed{\ln(v) + v = \frac{c}{x}}\text{; that is } \boxed{\ln\left(\frac{y}{x}\right) + \frac{y}{x} = \frac{c}{x}}$$

As $y(1) = e$, we get $1 + e = \frac{c}{1} \rightarrow \boxed{c = 1+e}$

Therefore, the unique sol of the IVP is given by: $\ln\left(\frac{y}{x}\right) + \frac{y}{x} = \frac{e+1}{x}$

Exercise 4. An object at 46°C was put into a refrigerator. Ten minutes later the temperature of the object was 39°C and ten minutes after that its temperature was 33°C. Find the temperature inside the refrigerator.

solution.

By NLCW, the DE governing this model is

$\theta'(t) = k(\theta(t) - A)$, where $\theta(t)$ is the temperature of the object at time t and A is the temperature of the refrigerator.

The solution of the DE is $\theta(t) = A + ce^{kt}$.

Now, as $\theta(0) = 46$, $\theta(10) = 39$ and $\theta(20) = 33$, we get

① $A + ce^{10k} = 39$, ② $A + ce^{20k} = 33$, and $A + C = 46$ ③.

By performing ③ − ① and ① − ②, we obtain:

$c(1 - e^{10k}) = 7$, $ce^{10k}(1 - e^{10k}) = 6$

$\Rightarrow 7e^{10k} = 6 \longrightarrow k = \frac{1}{10}\ln\left(\frac{6}{7}\right)$.

Hence $c\left(1 - e^{\ln\left(\frac{6}{7}\right)}\right) = 7 \longrightarrow c\left(1 - \frac{6}{7}\right) = 7$

$\longrightarrow \frac{c}{7} = 7 \longrightarrow \underline{c = 49}$

Consequently, $A = 46 - C = 46 - 49 = \underline{-3}$.