KFUPM $\qquad$ T162

Dept. Math & Stat $\qquad$ Math 202 / Sec-13

TET N° 1

Exercise 1: Show that $y_1 = 1$ and $y_2 = 3x^2+1$ are solutions of the IVP:

$$\left(x\frac{dy}{dx} = 2(y-1),\ y(0)=1\right)$$ on the interval

$I = (-\infty, \infty)$.

Solution.

- As $y_1' = 0$, we have $x y_1' = 0 = 2(y_1-1)$ and $y_1(0) = 1$, the function $y_1 = 1$ is a solution of the given IVP.
- Also, $y_2' = 6x$, we have $x.y_2' = 6x^2 = 2(3x^2+1-1)$
$= 2(y_1 - 1)$,

$y_2(0) = 3\times 0 + 1 = 1$.
So y_2 is a solution of the given IVP.

The standard form of the DE is $\frac{dy}{dx} = \frac{2}{x}(y-1)$.
As the function $f(x,y) = \frac{2}{x}(y-1)$ is not continuous on a rectangular region containing the point $(x_0,y_0) = (0,1)$, the theorem of existence and uniqueness cannot be applied.

Exercise 2. Show that the IVP:
$((4-y^2)y' = x^2,\ y(2)=1)$ has a unique solution on an appropriate interval centred at 2.

Solution: The standard form of the DE is:

$$y' = \frac{x^2}{4-y^2} = f(x,y)$$

$f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on the region of $\mathbb{R}^2$ given by $R = \{(x,y)\in \mathbb{R} \mid y \neq \pm 2\}$. As the region R contains a rectangular region containing $(x_0,y_0) = (2,1)$.

So the IVP has a unique solution.

Exercise 3.

① Solve the DE : $(y^2-1)\,dx-(x^2-1)\,dy=0$

② Find all the constant singular solutions of the DE.

③ Solve the IVP:

$$\left((y^2-1)\,dx-(x^2-1)\,dy=0,\ y(0)=1\right)$$

and find the interval of definition of the solution.

Solution.

① Here the DE is separable.

• Separating variables: $\dfrac{dx}{x^2-1}=\dfrac{dy}{y^2-1}$

• Integrating.

$$\int\frac{dx}{x^2-1}=\int\frac{dy}{y^2-1}$$

$$\rightarrow \int\frac{1}{2}\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\int\frac{1}{2}\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy$$

$$\rightarrow \int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\int\left(\frac{1}{y-1}-\frac{1}{y+1}\right)dy$$

$$\rightarrow \ln|x-1|-\ln|x+1|=\ln|y-1|-\ln|y+1|+c_2$$

$$\rightarrow \ln\left|\frac{x-1}{x+1}\right|=\ln\left|\frac{y-1}{y+1}\right|+c_1$$

$$\rightarrow \left|\frac{x-1}{x+1}\right|=\left|\frac{y-1}{y+1}\right|e^{c_1}\longrightarrow \frac{y-1}{y+1}=c\left(\frac{x-1}{x+1}\right),\ \text{where } c\in\mathbb{R}$$

So $(y-1)(x+1)=c(y+1)(x-1)$

$$\rightarrow y\left[(x+1)-c(x-1)\right]=x+1+c(x-1)=(1+c)x+1-c$$

$$\rightarrow y\left((1-c)x+c+1\right)=(c+1)x-c+1$$

$$\rightarrow \boxed{y=\frac{(c+1)x-c+1}{(1-c)x+c+1}}$$

② The constant solutions $y=k$ are given by:

$$k^2-1-(x^2-1)\frac{dy}{dx}=0\ \leftrightarrow\ k^2=1\ \Leftrightarrow\ (k=1)\text{ or }(k=-1)$$

Thus, $y_1=1$, $y_2=-1$ are the constant solutions of the DE.

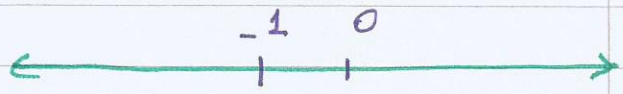
Sol Ex 3 (3)

The one-parameter family of solutions associated with the DE is $y = \dfrac{(c+1)x - c + 1}{(1-c)x + c + 1}$.

IC : $y(0) = 1 \iff 1 = \dfrac{-c+1}{c+1} \iff c+1 = -c+1$
$\iff c = 0$.

Therefore $y = \dfrac{x+1}{x+1} = 1$ is the unique solution of the given IVP:

$\mathbb{R}$:



The largest open interval containing 0 and not containing -1 is $I = (-1, \infty)$: the interval of validity of y.

$y_1 = 1$ is a member of the previous one-parameter family of solutions.

However $y_2 = -1$ is not a member; indeed, if, $\forall x$ such that $(1-c)x + c+1 \neq 0$.

$$-1 = \frac{(c+1)x - c+1}{(1-c)x + c+1} \iff (-1+c)x - c - 1 = (c+1)x - c + 1$$

$$\iff -x + \cancel{cx} - \cancel{c} - 1 = \cancel{cx} + x - \cancel{c} + 1$$

$$\iff x + 1 = 0$$

$$\iff \underline{x = -1}$$

So the solution is then not defined for $x \neq -1$.

It follows that $y_2 = -1$ is a singular solution.

Exercise 4.

Solve the DE: $x^2 y' + x(x+2)y = e^x$

Identify transient terms of the solution (i.e., the terms tending to 0 as x goes to ∞).

Solution.

Here, the DE is linear, its standard form is

$$\frac{dy}{dx} + \frac{x+2}{x} y = x^{-2} e^x$$

- $u(x) = e^{\int \frac{x+2}{x} dx} = e^{\int (1+\frac{2}{x}) dx} = e^{x + 2\ln x} = x^2 e^x$ is an integrating factor of the DE.

- So, multiplying by $u(x) = x^2 e^x$ both sides of the standard form, we get:

$$\frac{d}{dx}\left[y\, x^2 e^x\right] = (x^{-2} e^x)\, x^2 e^x = e^{2x}$$

$$\rightarrow y\, x^2 e^x = \frac{1}{2} e^{2x} + c \rightarrow y = \frac{1}{2} x^{-2} e^x + \frac{c}{x^2 e^x}$$

The transient term is $\frac{c}{x^2 e^x}$.

Exercise 5. Find a solution of the form

$y = ae^x + bx$ of the DE:

$$y' + xy^2 = xe^{2x} + 2x^2e^x + e^x + x^3 + 1$$

Solution.

If $y = ae^x + bx$, then $y' = ae^x + b$.

So $y = ae^x + bx$ is a solution iff

$$ae^x + b + x(ae^x + bx)^2 = xe^{2x} + 2x^2e^x + e^x + x^3 + 1$$

$$\Leftrightarrow ae^x + b + a^2xe^{2x} + bx^3 + 2abx^2e^x = xe^{2x} + 2x^2e^x + e^x + x^3 + 1$$

$\Leftrightarrow$ $(1 = a^2)$ and $(2 = 2ab)$ and $(1 = a)$ and $(b = 1)$

Therefore $\underline{a = b = 1}$.

It follows that $y = e^x + x$ is a solution of the given DE.