King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam II - Term 162

Duration: 120 minutes

Name: KEY ID Number:

Section Number: Serial Number:

Class Time: Instructor's Name:

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 8 pages of problems (Total of 13 Problems)
5. **DE means differential equations.**

Question # Number	Points	Maximum Points
1		6
2		5
3		6
4		7
5		6
6		8
7		8
8		8
9		7
10		5
11		12
12		14
13		8
Total		100

1. **[6 points]** Given that $y = c_1 + c_2 x^3$ is a two-parameter family of solutions of the DE $xy'' - 2y' = 0$ on $(-\infty, \infty)$

(a) Show that no constants c_1, c_2 can be found so that $y = c_1 + c_2 x^3$ satisfies the initial conditions $y(0) = 1,\ y'(0) = 2$

(b) Explain why this does not violate the theorem of existence and uniqueness of a solution of an IVP.

(a) $y = c_1 + c_2 x^3 \Rightarrow y' = 3c_2 x^2$ 1pt

$y(0) = 1 \Rightarrow c_1 = 1$ 1pt

$y'(0) = 2 \Rightarrow 2 = 0$ which is not possible 1pt

Thus, no constants can be found. 1pt

(b) Since the coefficient of y'' is 0 at $x = 0$, then the mentioned Theorem is not violated. 2pts

2. **[5 points]** Without the use of the Wronskian, show that the functions $f_1(x) = (\cos^2 x)(1 + \tan x)^2$, $f_2(x) = 7$, and $f_3(x) = \sin 2x$ are linearly dependent on the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

$f_1(x) = (\cos x + \sin x)^2 = 1 + \sin 2x$ 1pt

So, $f_1(x) - \frac{1}{7} f_2(x) - f_3(x) = 1 + \sin 2x - 1 - \sin 2x = 0$

2pts

Therefore, f_1, f_2, and f_3 are linearly dependent

2pts

3. **[6 points]** Without solving the DE, show that the function $y_1 = x$ and $y_2 = x^{-1}$ form a fundamental set of solutions of the DE

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0,\ x > 0.$$

$y_1 = x \Rightarrow y_1' = 1$ and $y_1'' = 0 \Rightarrow y_1'' + \frac{1}{x}y_1' - \frac{1}{x^2}y_1 = 0$

$y_2 = x^{-1} \Rightarrow y_2' = -x^{-2}$ and $y_2'' = 2x^{-3} \Rightarrow y_2'' + \frac{1}{x}y_2' - \frac{1}{x^2}y_2 = 0$

Therefore, $y_1 = x$ and $y_2 = x^{-1}$ are solutions of the given DE (2pts)

$W(x, x^{-1}) = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1} \neq 0$ for all $x > 0$ (2pts)

Thus, $y_1 = x$ and $y_2 = x^{-1}$ are linearly independent for all $x > 0$ (1pt)

Therefore, y_1 and y_2 form a fundamental set of solutions of the given DE. (1pt)

4. **[7 points]** Given that $y_1 = \frac{1}{4}\sin 2x$ and $y_2 = \frac{1}{4}x - \frac{1}{8}$ are, respectively, particular solutions of the DEs $L(y) = \cos 2x$ and $L(y) = x$, where L is a linear differential operator. Find a particular solution of the DE $L(y) = 11\,x - 12\cos 2x$.

$L(\frac{1}{4}\sin 2x) = \cos 2x$

$\Rightarrow L(-3\sin 2x) = -12\cos 2x$ (2pts)

and $L(\frac{1}{4}x - \frac{1}{8}) = x$

$\Rightarrow L(\frac{11}{4}x - \frac{11}{8}) = 11x$ (2pts)

$\Rightarrow L(\frac{11}{4}x - \frac{11}{8}) + L(-3\sin 2x) = L(\frac{11}{4}x - \frac{11}{8} - 3\sin 2x)$

$= 11x - 12\cos 2x$ (2pts)

$\Rightarrow y_p = \frac{11}{4}x - \frac{11}{8} - 3\sin 2x$ is a particular solution of

$L(y) = 11x - 12\cos 2x$ (1pt)

5. **[6 points]** Given that $y_1 = \dfrac{\cos x}{\sqrt{x}}$ is a solution of the DE
$x^2y'' + xy' + \left(x^2 - \dfrac{1}{4}\right) y = 0$ on $\left(0, \dfrac{\pi}{2}\right)$. Find a second linearly independent solution.

Form the standard form of the equation (1pt)

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right) y = 0$$

Then, we use the formula (1pt)

$$y_2 = y_1(x) \int \frac{e^{-\int p(x)dx}}{(y_1(x))^2}\, dx, \quad \text{with } p(x) = \frac{1}{x} \quad \text{(1pt)}$$

$$\Rightarrow y_2 = \frac{\cos x}{\sqrt{x}} \int \frac{\frac{1}{x}}{\frac{\cos^2 x}{x}}\, dx = \frac{\cos x}{\sqrt{x}} \int \sec^2 x\, dx$$

$$= \frac{\cos x}{\sqrt{x}} \cdot \tan x = \frac{\sin x}{\sqrt{x}} \quad \text{(3 pts)}$$

6. **[8 points]** Find the general solution of the DE

$$y''' - 5y'' + 12y' - 8y = 0.$$

The auxiliary equation is $m^3 - 5m^2 + 12m - 8 = 0$ (1pt)

has $m = 1$ as a root (1pt)

1	1	-5	12	-8
		1	-4	8
	1	-4	8	0

$\Rightarrow (m-1)(m^2 - 4m + 8) = 0$

The roots of $m^2 - 4m + 8 = 0$ are

$$m = \frac{4 \pm \sqrt{-16}}{2} = 2 \pm 2i$$

Therefore, the general solution is

$$y = c_1 e^{x} + c_2 e^{2x} \cos 2x + c_3 e^{2x} \sin 2x$$

7. **[8 points]** Find a homogeneous linear DE of a minimal order such that $y = x^2 - xe^{2x}$ is a solution. **What is its general solution**?

The auxiliary equation must have the roots 0 (of multiplicity 3), and 2 (of multiplicity 2) (2pts)

That is, the auxiliary equation can be taken as (2pts)

$m^3(m-2)^2 = 0$

Thus, a DE having the given solution is (2pts)

$D^3(D-2)^2 y = 0$

whose general solution is

$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{2x} + c_5 x e^{2x}$ (2pts)

8. **[8 points]** Find a linear differential operator that annihilates the function $g(x) = (e^x + x\,e^{-x})^2 + x^2 \cos 4x$.

$g(x) = e^{2x} + 2x + x^2 e^{-2x} + x^2 \cos 4x$ (1pt)

Annihilators $D-2$ (1pt) D^2 (1pt) $(D+2)^3$ (1pt) $(D^2+16)^3$ (2pts)

So, the differential annihilator is

$(D-2) . D^2 (D+2)^3 (D^2+16)^3$ (2pts)

9. **[7 points]** Given that $y_1 = x$ is a solution of the DE

$$x^2\,y'' - (x^2+2x)\,y' + (x+2)\,y =$$ [redacted]^{x}, $x > 0$.

Use reduction of order method to reduce the given DE into an equation of the first order.**(Do not solve the new equation)**

Let $y = x\cdot u(x) \Rightarrow y' = xu' + u$ and

$y'' = xu'' + 2u'$ (2 pts)

Therefore, the DE becomes

$$x^3u'' + \cancel{2x^2u'} - x^3u' - \cancel{x^2u} - \cancel{2x^2u'} - \cancel{2xu}$$
$$+\cancel{x^2u} + \cancel{2xu} = 0$$

(2 pts)

$\Rightarrow x^3u'' - x^3u' = 0$

$\Rightarrow u'' - u' = 0$

Let $W = u'$ (1 pt)

$\Rightarrow W' - W = 0$ (2 pts)

is a first-order DE as required

10. **[5 points]** Use a suitable substitution to transform the DE

$$x^2y'' - y = \cos(\ln x),\ x > 0,$$

into a linear DE with constant coefficients.**(Do not solve the new equation)**

Let $x = e^t \Rightarrow t = \ln x$,

$x\dfrac{dy}{dx} = \dfrac{dy}{dt}$ and $x^2\dfrac{d^2y}{dx^2} = \dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}$ (3 pts)

Thus, the DE becomes

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + y = \cos t$$

which is the required DE (2 pts)

11. **[12 points]** Find the general solution of the DE

$$y'' - 2y' + y = x + e^{-2x}.$$

The auxiliary equation for the associated homogeneous eqn is $m^2 - 2m + 1 = (m-1)^2 = 0$ (1pt)

So, the complementary function is (2pts)

$$y_c = c_1 e^x + c_2 x e^x$$

Now, since $D^2 x = 0$ and $(D+2)e^{-2x} = 0$

we apply the differential operator $D^2(D+2)$ to both sides of the given DE: (2pts)

$$D^2(D+2)(D-1)^2 y = 0$$

whose auxiliary equation is (2pts)

$$y = \boxed{c_1 e^x + c_2 x e^x} + c_3 + c_4 x + c_5 e^{-2x}$$

From which we get the form of y_p:

$$y_p = A + Bx + Ce^{-2x}$$ (1pt)

Substituting y_p in the given DE, we get

$$(4ce^{-2x}) - 2B + 4ce^{-2x} + A + Bx + Ce^{-2x} = x + e^{-2x}$$

$$\Rightarrow (A - 2B) + Bx + 9Ce^{-2x} = x + e^{-2x}$$

$$\Rightarrow B = 1,\ C = \frac{1}{9}, \text{ and } A = 2 \rightarrow$$ (3pts)

$$y_p = 2 + x + \frac{1}{9}e^{-2x}$$

$$\Rightarrow \text{G.S is } y = y_c + y_p = c_1 e^x + c_2 x e^x + 2 + x + \frac{1}{9}e^{-2x}$$ (1pt)

12. **[14 points]** Find the general solution of the DE

$$xy'' - 4xy = e^{2x}, \ x > 0$$

We first put the DE in the standard form

$$y'' - 4y = \frac{e^{2x}}{x} \quad \text{1pt}$$

The roots of the auxiliary eq: $m^2 - 4 = 0$

are $m = \pm 2 \Rightarrow$ The solutions of the homogeneous are $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ 2pts

We use the method of Variation of Parameters to find y_p:

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -4, \quad W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \frac{e^{2x}}{x} & -2e^{-2x} \end{vmatrix} = -\frac{1}{x}$$

$$\text{and } W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \frac{e^{2x}}{x} \end{vmatrix} = \frac{e^{4x}}{x} \quad \text{3pts}$$

$$u_1' = \frac{W_1}{W} = \frac{1}{4x}, \quad u_2' = \frac{W_2}{W} = -\frac{e^{4x}}{4x} \quad \text{2pts}$$

$\Rightarrow u_1 = \frac{1}{4}\ln x, \quad u_2 = -\frac{1}{4}\int_{x_0}^{x} \frac{e^{4t}}{t}\,dt$ where x_0 is any point in the interval of the solution 2pts

Thus $y_p = \frac{1}{4}e^{2x}\ln x - \frac{1}{4}e^{-2x}\int_{x_0}^{x} \frac{e^{4t}}{t}\,dt$ 2pts

G.S. $y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{4}e^{2x}\ln x - \frac{1}{4}e^{-2x}\int_{x_0}^{x} \frac{e^{4t}}{t}\,dt$ 2pts

13. **[8 points]** Use the substitution $y = x^m$ to solve the DE
$x^3y''' + 3x^2y'' + 5xy' = 0$, $x > 0$.

$y = x^m$. The auxiliary eq$\underline{n}$ is

$$m(m-1)(m-2) + 3m(m-1) + 5m = 0$$ 2pts

$$\Rightarrow m(m^2+4) = 0$$

$$\Rightarrow m = 0, \quad m = \pm 2i$$ 3pts

The G.S. is

$$y = c_1 + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x)$$ 3pts