King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

MATH 202 - Exam I - Term 162

Duration: 120 minutes

Name:______________________ ID Number:______________________

Section Number:______________________ Serial Number:______________________

Class Time: ______________________ Instructor's Name:______________________

Instructions:

1. Calculators and Mobiles are not allowed.
2. Write legibly.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 9 pages of problems (Total of 10 Problems)
5. **DE means differential equations.**

Question # Number	Points	Maximum Points
1		11
2		6
3		6
4		11
5		12
6		12
7		8
8		12
9		12
10		10
Total		100

1. (a) **[5 points]** Verify that $x = 1 + \frac{1}{27}(3t^2 + c)^3$ is a one-parameter family of solutions of the DE
$$\frac{dx}{dt} = 6t(x-1)^{2/3}$$

$x = 1 + \frac{1}{27}(3t^2+c)^3 \Rightarrow \frac{dx}{dt} = \frac{1}{9}(3t^2+c)^2(6t) = \frac{2}{3}t\,(3t^2+c)^2$ — ① (2pts)

and $(x-1)^{2/3} = \left[\frac{1}{27}(3t^2+c)^3\right]^{2/3} = \frac{1}{9}(3t^2+c)^2$ — ② (2pts)

① and ② $\Rightarrow \frac{dx}{dt} = 6t(x-1)^{2/3}$ (1pt)

(b) **[6 points]** Find a singular constant solution of the DE given in (a) **(Give a reason to your answer.)**

Constant Solutions of the DE are $x = k$ (1pt)

such that $0 = 6t(k-1)^{2/3}$ for every real number t

that is $k = 1$. (1pt)

It follows that the unique constant solution is $x = 1$ — (1pt)

If $x=1$ is a member of the given one-parameter family of solutions, then there is a real number c such that $1 = 1 + \frac{1}{27}(3t^2+c)^3$ for every real number t which is impossible (2pts)

$\therefore$ $x=1$ is a singular constant solution (1pt)

2. **[6 points]** Given that $2x^2 - y^2 - 2x = c$ is a one-parameter family of solutions of the DE $y\dfrac{dy}{dx} = 2x - 1$.

   Find an explicit solution which satisfies the initial condition $y(1) = -2$.

By substituting $x=1$ and $y=-2$ in $2x^2-y^2-2x=c$, we get $c=-4$ (1pt)

$\Rightarrow 2x^2-y^2-2x=-4$ ①

By solving ① with respect to y, we find the explicit Solutions $y=\sqrt{2x^2-2x+4}$ and $y=-\sqrt{2x^2-2x+4}$ (2pts)

obviously among these, only $y=-\sqrt{2x^2-2x+4}$ satisfies the condition $y(1)=-2$. (2pts)

Hence, the required explicit solution is

$y=-\sqrt{2x^2-2x+4}$ (1pt)

3. **[6 points]** Determine whether the theorem on the existence of a Unique Solution guarantees that the initial value problem $\dfrac{dy}{dx} = xy^{2/3}$, $y(1) = 0$ possesses a unique solution. **[Do not solve the differential equation]**

$f(x,y) = xy^{2/3}$ which is continuous for all (x,y) (1pt)

$\dfrac{\partial f}{\partial y} = \dfrac{2}{3}xy^{-1/3}$ which is continuous for all $y \neq 0$ (1pt)

Therefor f and $\dfrac{\partial f}{\partial y}$ are not continuous in any rectangular region that contains the point $(1,0)$ in its interior -- (2 pts)

So that the theorem does not guarantee that the IVP possesses a unique solution

(2pts)

4. [**11 points**] Find all explicit solutions of the DE $\dfrac{dy}{dx} = \dfrac{xy^2 + x}{x^2y + y}$, $y \neq 0$.

$$\left.\begin{aligned}\frac{dy}{dx} &= \frac{xy^2+x}{x^2y+y} = \frac{x(y^2+1)}{y(x^2+1)} \\ &= \left(\frac{x}{x^2+1}\right)\left(\frac{y^2+1}{y}\right)\end{aligned}\right\}$$

2 pts

Therefore, the DE is seperable

So $\dfrac{y}{y^2+1}\,dy = \dfrac{x}{x^2+1}\,dx$

2 pts

Integrating both sides, we obtain

$$\int \frac{y}{y^2+1}\,dy = \int \frac{x}{x^2+1}\,dx$$

1 pt

So $\ln(y^2+1) = \ln(x^2+1) + \ln c$, $c > 0$

2 pts

Then $y^2+1 = c(x^2+1)$

2 pts

and hence $y = \pm\sqrt{c(x^2+1) - 1}$

2 pts

5. **[12 points]** Solve the Initial-Value Problem

$$x^2(x-2)\,\frac{dy}{dx}+x(x-2)\,y=2,\;\; y(1)=1,$$

and give **the interval of definition** of the solution.

The given DE is linear in y
Dividing by $x^2(x-2)$, the standard form of the DE is $\frac{dy}{dx}+\frac{1}{x}y=\frac{2}{x^2(x-2)}$, $x\neq 0,2$ (2pts)

The functions $P(x)=\frac{1}{x}$ and $f(x)=\frac{2}{x^2(x-2)}$ are continuous on $(-\infty,0)$, $(0,2)$, and $(2,\infty)$. We solve the equation on $(0,2)$ because of the initial condition $y(1)=1$. An integrating factor $=e^{\int\frac{1}{x}dx}=e^{\ln x}=x$ (2pts)

Multiply the standard form equation by x, we get

$$\frac{d}{dx}(xy)=\frac{2}{x(x-2)},\quad x\in(0,2)$$ (2pts)

$$\Rightarrow xy=\int\frac{2}{x(x-2)}\,dx+C=\int\left(\frac{1}{x-2}-\frac{1}{x}\right)dx+C$$

$$=\ln(2-x)-\ln x+C,\quad x\in(0,2)$$ (3pts)

$$\Rightarrow y=\frac{1}{x}\ln\left(\frac{2-x}{x}\right)+\frac{C}{x}$$

$y(1)=1\Rightarrow C=1$ (1pt)

$\Rightarrow$ The solution of the IVP is

$$y=\frac{1}{x}\ln\left(\frac{2-x}{x}\right)+\frac{1}{x}$$

interval of definition $(0,2)$ (2pts)

6. **[12 points]** Verify that the DE is exact, and then solve it:

$$\left(\frac{e^y}{x} - \frac{y}{1+x^2}\right) dx + (e^y \ln x - \tan^{-1} x + 2)\, dy = 0$$

Set $M = \frac{e^y}{x} - \frac{y}{1+x^2}$ and $N = e^y \ln x - \tan^{-1} x + 2$

Since $\frac{\partial M}{\partial y} = \frac{e^y}{x} - \frac{1}{1+x^2} = \frac{\partial N}{\partial x}$, the equation is exact $\Big\}$ 3 pts

From $\frac{\partial f}{\partial x} = M = \frac{e^y}{x} - \frac{y}{1+x^2}$, we find by integrating with respect to x, $f(x,y) = e^y \ln x - y \tan^{-1} x + g(y)$ 2 pts

From $\frac{\partial f}{\partial y} = N$, we obtain

$e^y \ln x - \tan^{-1} x + g'(y) = e^y \ln x - \tan^{-1} x + 2$ $\Big\}$ 3 pts

Hence, $g'(y) = 2$ and we may take $g(y) = 2y$ 2 pts

So $f(x,y) = e^y \ln x - y \tan^{-1} x + 2y$, and a one-parameter family of solutions

$e^y \ln x - y \tan^{-1} x + 2y = c$ $\Big\}$ 2 pts

7. **[8 points]** Show that the differential equation $(3x^2y - 8x)\,y' = 4y - 2xy^2$ is not exact. Then find an appropriate integrating factor which can be used to make the differential equation exact. **Find the exact equation but do not solve it.**

We write the equation as

$$(4y - 2xy^2)dx + (8x - 3x^2y)\,dy = 0$$

hence, $M(x,y) = 4y - 2xy^2$ and $N(x,y) = 8x - 3x^2y$

Since $M_y = 4 - 4xy \neq N_x = 8 - 6xy$,

So the equation is not exact

(3 pts)

The function $\frac{M_y - N_x}{N} = \frac{-4 + 2xy}{8x - 3x^2y}$ depends on x and y

while the function $\frac{N_x - M_y}{M} = \frac{4 - 2xy}{4y - 2xy^2} = \frac{1}{y}$ depends

on y only. So, there is an integrating factor

(3 pts)

$$\mu(y) = e^{\int \frac{1}{y} dy} = y$$

We multiply the given DE by y to obtain

an exact equation

$$(4y^2 - 2xy^3)\,dx + (8xy - 3x^2y^2)\,dx = 0$$

(2 pts)

8. **[12 points]** Solve the DE $\frac{dy}{dx} = y\,(xy^3 - 1)$

The DE can be written as

$$\left.\begin{array}{ll} \frac{dy}{dx} + y = x y^4 & (i) \\ \text{or } \; y^{-4}\frac{dy}{dx} + y^{-3} = x & (ii) \\ \text{which is of Bernoulli's Type} & \end{array}\right\} \text{2 pts}$$

$$\text{Let } u = y^{-3} \Rightarrow \frac{du}{dx} = -3y^{-4}\frac{dy}{dx}, \quad \text{2 pts}$$

then equation (ii) becomes

$$\left.\begin{array}{ll} -\frac{1}{3}\frac{du}{dx} + u = x & \\ \Rightarrow \frac{du}{dx} - 3u = -3x & (iii) \end{array}\right\} \text{2 pts}$$

An integrating factor for the linear equation (iii) is

$e^{\int -3\,dx} = e^{-3x}$. 1 pt Multiplying equation (iii)

$$\left.\begin{array}{l} \text{by } e^{-3x}\text{, we get } \frac{d}{dx}\left(u e^{-3x}\right) = -3x e^{-3x} \\ \Rightarrow u e^{-3x} = \int -3x e^{-3x}\,dx + C \end{array}\right\} \text{2 pts}$$

$$\Rightarrow y^{-3} e^{-3x} = -3\left[-\frac{1}{3}x e^{-3x} - \frac{1}{9}e^{-3x}\right] + C \quad \text{2 pts}$$

So a solution of the given DE is

$$y^{-3} = x + \frac{1}{3} + C e^{3x} \quad \text{1 pt}$$

$$\text{or } y = \left(x + \frac{1}{3} + C e^{3x}\right)^{-1/3}$$

9. Consider the DE $\frac{dy}{dx} = \frac{x+y-1}{x-y-2}$

(a) [**6 points**] For which values of a and b the change of variables $x = t + a$ and $y = z + b$ transforms the DE to $\frac{dz}{dt} = \frac{t+z}{t-z}$

$X = t + a$ and $y = z + a \Rightarrow \frac{dz}{dt} = \frac{dy}{dx}$ 1 pt

So the DE transformed into 2 pts

$$\frac{dz}{dt} = \frac{t+a+z+b-1}{t+a-z-b-2} = \frac{t+z+a+b-1}{t-z+a-b-2}$$

We shall choose a and b such that

$a+b-1=0$ and $a-b-2=0$

Thus $a = \frac{3}{2}$ and $b = -\frac{1}{2}$ 3 pts

(b) [**6 points**] Find a suitable substitution to transform the resulting DE obtained in (a) to a separable equation. [**Do not solve the equation.**]

The DE $\frac{dz}{dt} = \frac{t+z}{t-z}$ can be written as

$$\frac{dz}{dt} = \frac{1+\frac{z}{t}}{1-\frac{z}{t}} = f\left(\frac{z}{t}\right) \quad (c)$$ 1 pt

Thus it is a homogeneous equation

Let $z = ut \Rightarrow \frac{dz}{dt} = u + t\frac{du}{dt}$ 2 pts

So DE (c) transformed into

$$u + t\frac{du}{dt} = \frac{1+u}{1-u}$$ 1 pt

from which we get a separable DE

$$\frac{du}{dt} = \left(\frac{1}{t}\right)\left(\frac{1+u^2}{1-u}\right)$$ 2 pts

10. A thermometer is taken form a room where the temperature is 20° C to the outside where the temperature from 25° C. After one minute, the thermometer reading is 23° C.

(a) [**8 points**] What will be the exact reading of the thermometer after one more minute?

(b) [**2 points**] When exactly will the reading of the thermometer be 24.68° C?

(a) $\left.\begin{aligned} \frac{dT}{dt} &= k(T-25); \\ T(0) &= 20. \end{aligned}\right\}$ 2pts

The solution of the initial value problem above is

$$T = 25 - 5e^{kt}$$ 2pts

The condition $T(1) = 23$ gives

$$2 = 5e^{k}$$

$$\Rightarrow e^{k} = 0.4$$ 2pts

At $t = 2$; $T = 25 - 5e^{2k} = 25 - 5 * (0.4)^2$

$= 25 - 0.8 = 24.2°C$ 2pts

(b) When $T = 24.68$,

$$24.68 = 25 - 5 * (0.4)^t$$

$$\Rightarrow .064 = (0.4)^t$$

$\Rightarrow t = 3$ minutes 2pts