King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics

# MATH 201 - Term 162 - Exam II

Duration: 120 minutes

Key

Name:________________ ID Number:________________

Section Number:________________ Serial Number:________________

ClassTime: ________________ Instructor's Name:________________

**Instructions:**

1. Calculators and Mobiles are not allowed.
2. Write neatly and legibly. You may lose points for messy work.
3. Show all your work. No points for answers without justification.
4. Make sure that you have 6 pages of problems ( Total of 10 Problems)

| Question Number | Points | Maximum Points |
|---|---|---|
| 1 | | 18 |
| 2 | | 7 |
| 3 | | 12 |
| 4 | | 6 |
| 5 | | 8 |
| 6 | | 8 |
| 7 | | 10 |
| 8 | | 13 |
| 9 | | 9 |
| 10 | | 9 |
| Total | | 100 |

1. Consider the lines

$$L1: x = -4t,\ \ y = 1,\ \ z = 4 + t,\ \ -\infty < t < \infty,$$

$$L2: x = 2s - 4,\ \ y = s + 1,\ \ z = 5 - s,\ \ -\infty < s < \infty.$$

a) (6-points) Find, if it exists, the point at the intersection of $L1$ and $L2$.

1pt $\begin{cases} -4t = 2s - 4 \\ 1 = s + 1 \\ 4 + t = 5 - s \end{cases}$

Using the first two equations in this system, we find that $t = 1$ and $s = 0$. 2 pts

We verify that these values satisfy the 3rd eqn:

$4 + 1 = 5 - 0$. 1 pt

Then the lines $L_1$ and $L_2$ intersect.

The point of intersection is $(-4, 1, 5)$ which can be found using $L_1$ and $t = 1$ or $L_2$ and $s = 0$. 2 pts

b) (7-points) Find the symmetric equations of the line passing through the point $(1, 2, 3)$ and perpendicular to $L1$ and $L2$.

the direction vector of $L_1$ is $d_1 = \langle -4, 0, 1 \rangle$ 1 pt

the direction vector of $L_2$ is $d_2 = \langle 2, 1, -1 \rangle$. 1 pt

the direction of the line perpendicular to both $L_1$ and $L_2$ is

$$d_1 \times d_2 = \begin{vmatrix} i & j & k \\ -4 & 0 & 1 \\ 2 & 1 & -1 \end{vmatrix} = \langle -1, -2, -4 \rangle. \quad \text{2 pts}$$

the symmetric eqns are: $\frac{x-1}{-1} = \frac{y-2}{-2} = \frac{z-3}{-4}$ 3 pts

c) (5-points) Find an equation of the plane containing the lines $L1$ and $L2$.

$d_1 \times d_2$, as in part (b), is normal to the plane containing the lines $L_1$ and $L_2$. 1 pt

using the point of the intersection of $L_1$ and $L_2$ from part (a), an eqn of the plane is: 1 pt

$(-1)(x+4) + (-2)(y-1) + (-4)(z-5) = 0$ or $x + 2y + 4z = 18$ 3 pts

2. (7-points) Identify the quadric surface $x^2 + 4y^2 - 2x + 16y - 2z + 18 = 0$ and its traces on the planes $z = k$ for $k$ any real number.

$$x^2 - 2x + 1 + 4(y^2 + 4y + 4) - 2z + 18 = 1 + 16$$

$$(x-1)^2 + 4(y+2)^2 = 2z - 1 \Rightarrow \underbrace{\frac{(x-1)^2}{2} + 2(y+2)^2 = z - \frac{1}{2}}$$

this is an elliptical paraboloid 2pts

the equations of the traces are $\underbrace{\frac{(x-1)^2}{2} + 2(y+2)^2 = k - \frac{1}{2}}_{1pt}$, $k \geq \frac{1}{2}$

1pt

if $\boxed{k = \frac{1}{2}}$, then $\frac{(x-1)^2}{2} + 2(y+2)^2 = 0$. So the trace is the point $(1, -2, \frac{1}{2})$. 1pt

if $\boxed{k > \frac{1}{2}}$, then the traces are ellipses. 1pt

1pt

3. Let $f(x, y) = \dfrac{1}{\ln(e^2 - x^2 - 4y^2)}$.

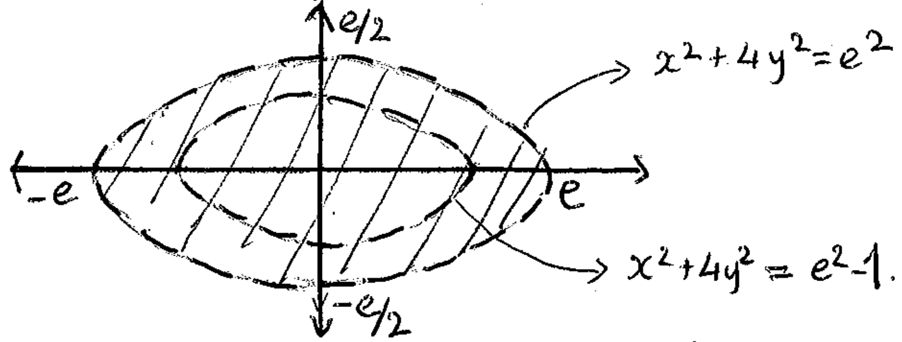
(a) (8-points) Find and sketch the domain of $f$.

The domain of $f$ is all $(x, y)$ in $\mathbb{R}^2$ so that

$$\underbrace{e^2 - x^2 - 4y^2 > 0}_{2pts} \quad \text{and} \quad \underbrace{e^2 - x^2 - 4y^2 \neq 1}_{1pt}$$

or

$$\text{Domain} = \{(x, y) \in \mathbb{R}^2 \mid e^2 > x^2 + 4y^2 \text{ and } x^2 + 4y^2 \neq e^2 - 1\}.$$




* sketch of ellipses 2pts each.

* region 1pt

* including ellipse in sketch −1pt for each ellipse

(b) (4-points) Find and identify the level curve $f(x, y) = \frac{1}{2}$.

$$\frac{1}{\ln(e^2 - x^2 - 4y^2)} = \frac{1}{2} \Rightarrow \ln(e^2 - x^2 - 4y^2) = 2 \Rightarrow e^2 - x^2 - 4y^2 = e^2$$

1pt

$$\Rightarrow x^2 + 4y^2 = 0$$ 1pt

So the level curve is the point $(0, 0)$. 2pts

4. (6-points) Find the limit or show that it does not exist.

$$\lim_{(x,y)\mapsto(0,0)} \frac{x^3+y^3}{\tan^{-1}(x^2+y^2)}$$

Using polar coordinates $x = r\cos\theta$ , $y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\tan^{-1}(x^2+y^2)} = \lim_{r\to 0} \frac{r^3(\cos^3\theta+\sin^3\theta)}{\tan^{-1}(r^2)}.$$ 2 pts

Using L'H rule $\lim_{r\to 0} \frac{r^3}{\tan^{-1}(r^2)} = \lim_{r\to 0} \frac{3r^2}{\frac{2r}{1+r^4}} = \lim_{r\to 0} \frac{3}{2}r(1+r^4) = 0$

2 pts 1 pt

As for any $\theta$, $\cos^3\theta + \sin^3\theta$ is bounded $\lim_{r\to 0} \frac{3}{2}r(1+r^4)(\cos^3\theta+\sin^3\theta) = 0$

So $\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{\tan^{-1}(x^2+y^2)} = 0$ 1 pt

5. (8-points) Determine whether $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$ is continuous at $(0,0)$ or not. Give reasons for your answer.

1 pt $f$ is continuous at $(0,0)$ whenever $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

2 pts along path $x=0$ $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} 0 = 0$

2 pts along path $y=x^2$ $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} f(x,x^2) = \lim_{x\to 0} \frac{x^4}{x^4+x^4} = \frac{1}{2}$

2 pts Then $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE

1 pt So $f$ is not continuous at $(0,0)$.

6. (8-points) Let $f(x,y,z) = xe^{xy}\sin^2 z$. Find $f_{xyz}(-1, 0, \pi/4)$.

2pts $f_x(x,y,z) = e^{xy}\sin^2 z + xye^{xy}\sin^2 z = (1+xy)e^{xy}\sin^2 z$

2pts $f_{xy}(x,y,z) = xe^{xy}\sin^2 z + (1+xy)xe^{xy}\sin^2 z = (2x+x^2y)e^{xy}\sin^2 z$

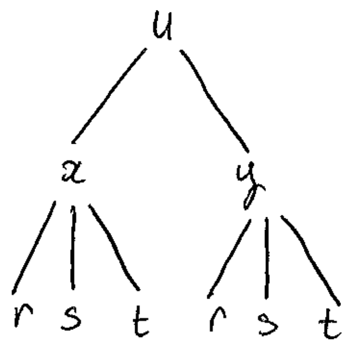
2pts $f_{xyz}(x,y,z) = (2x+x^2y)e^{xy}\,2\sin z\cos z = (2x+x^2y)e^{xy}\sin(2z)$

2pts $f_{xyz}(-1,0,\pi/4) = (-2)e^0\sin(\pi/2) = -2$

Note: An alternative solution using Clairaut's Thm is given at the end.

7. (10-points) Let $u = \sqrt{x^3 - y^3}$ where $x(r,s,t) = r(s+t)$ and $y(r,s,t) = \dfrac{r-s}{t}$. Use chain rule to calculate $\dfrac{\partial u}{\partial s}$ at $(r,s,t) = (3,-1,2)$.

$x(3,-1,2) = 3$ 1pt $\quad y(3,-1,2) = \dfrac{3-(-1)}{2} = 2$ 1pt



$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} \quad \text{1pt}$$

$$\frac{\partial u}{\partial x}(3,2) = \frac{1}{2\sqrt{x^3-y^3}}\,3x^2\Big|_{(3,2)} = \frac{27}{2\sqrt{19}} \quad \text{2pts}$$

$$\frac{\partial u}{\partial y}(3,2) = \frac{1}{2\sqrt{x^3-y^3}}(-3y^2)\Big|_{(3,2)} = \frac{-12}{2\sqrt{19}} \quad \text{2pts}$$

$$\frac{\partial x}{\partial s}(3,-1,2) = r\,\Big|_{(3,-1,2)} = 3 \quad \text{1pt}$$

$$\frac{\partial y}{\partial s}(3,-1,2) = \frac{-1}{t}\Big|_{(3,-1,2)} = \frac{-1}{2} \quad \text{1pt}$$

$$\frac{\partial u}{\partial s} = \frac{27}{2\sqrt{19}}\cdot 3 + \frac{12}{2\sqrt{19}}\cdot\frac{1}{2}$$

$$= \frac{87}{2\sqrt{19}} \quad \text{1pt}$$

8. Consider the surface given by the equation $\sin(xz) - 4\cos(yz) = 0$ and the point $P(\pi, -\pi/2, 1)$ on this surface.

   (a) (9-points) Find the parametric equations of the normal line to the above surface at the point $P$.

Consider the given surface as the level surface $f(x,y,z)=0$

where $f(x,y,z) = \sin(xz) - 4\cos(yz)$.

The direction of the normal line to this surface at $P$ is

$\nabla f(P) = \langle f_x(P), f_y(P), f_z(P) \rangle$.

$f_x(x,y,z) = z\cos(xz) \rightsquigarrow f_x(\pi, -\pi/2, 1) = -1.$ 2pts

$f_y(x,y,z) = 4z\sin(yz) \rightsquigarrow f_y(\pi, -\pi/2, 1) = -4$ 2pts

$f_z(x,y,z) = x\cos(xz) + 4y\sin(yz) \rightsquigarrow f_z(\pi, -\pi/2, 1) = \pi$ 2pts

Then $\nabla f(P) = \langle -1, -4, \pi \rangle$

Parametric equations of the normal line at $P$ are:

$x = \pi - t, \quad y = -\frac{\pi}{2} - 4t, \quad z = 1 - \pi t$ 3pts

   (b) (4-points) Find an equation of the tangent plane to the above surface at the point $P$.

$\nabla f(P) = \langle -1, -4, \pi \rangle$ is normal to the tangent plane at $P$. 1pt

An equation of this tangent plane is

$$-1\cdot(x-\pi) - 4\cdot\left(y - \frac{\pi}{2}\right) + \pi(z-1) = 0$$

or

$$x + 4y - \pi z = 2\pi$$

3pts

9. (9-points) Find the maximum rate of change of $f(x,y,z) = e^{xz-y^2}$ at the point $(-2,1,-1)$ and the unit vector along which it occurs.

$\nabla f(x,y,z) = \langle z e^{xz-y^2}, -2y e^{xz-y^2}, x e^{xz-y^2} \rangle$ 3 pts

$\nabla f(-2,1,-1) = \langle -e, -2e, -2e \rangle$ 1pt $|\nabla f(-2,1,-1)| = 3e$ 1pt

the maximum rate of change is $|\nabla f(-2,1,-1)| = 3e$ 2pts

the unit vector along which it occurs is $\dfrac{\nabla f(-2,1,-1)}{|\nabla f(-2,1,-1)|} = \dfrac{-1}{3}\langle 1,2,2 \rangle$

2pts

10. (9-points) Find the linearization of $f(x,y) = x^2y - \sqrt{xy} + 1$ at $(-1,-4)$ and use it to estimate the value of $f(-1.05,-3.96)$.

$f(-1,-4) = -4 - 2 + 1 = -5$ 1pt

$f_x(x,y) = 2xy - \dfrac{y}{2\sqrt{xy}} \rightsquigarrow f_x(-1,-4) = 8 - \dfrac{(-4)}{2 \cdot 2} = 9$ 2pts

$f_y(x,y) = x^2 - \dfrac{x}{2\sqrt{xy}} \rightsquigarrow f_y(-1,-4) = 1 - \dfrac{(-1)}{2 \cdot 2} = \dfrac{5}{4}$ 2pts

$L(x,y) = -5 + 9(x+1) + \frac{5}{4}(y+4).$ 2pts

$f(-1.05,-3.96) \approx L(-1.05,-3.96) = -5 + 9 \cdot (-0.05) + \frac{5}{4}(0.04)$

1pt

$= -5 - 0.45 + 0.05$

$= -5.4$ 1pt

An alternative Solution to Question 6

Using Clairaut's Thm, we can change the order of derivation from $f_{xyz}(-1,0,\pi/4)$ to $f_{zyx}(-1,0,\pi/4)$. Then

$$f_z(x,y,z) = x e^{xy} 2\sin z\cos z = x e^{xy}\sin(2z) \qquad \underline{\underline{2 \text{ pts}}}$$

$$f_{zy}(x,y,z) = x^2 e^{xy}\sin(2z) \qquad \underline{\underline{2 \text{ pts}}}$$

$$f_{zyx}(x,y,z) = 2x e^{xy}\sin(2z) + x^2 y e^{xy}\sin(2z) = (2x + x^2 y)e^{xy}\sin(2z) \qquad \underline{\underline{2 \text{ pts}}}$$

$$f_{zyx}(-1,0,\pi/4) = (-2)\cdot e^0 \sin(\pi/2) = -2 \qquad \underline{\underline{2 \text{ pts}}}$$