King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 550 – Linear Algebra (Term 161)

Final Exam

- Duration = **3 hours.**
- Each problem is worth **10 points.**

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(1) Let T be the linear operator on R^3, which is represented in the standard ordered basis by the matrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

(a) Find the minimal polynomial of T.
(b) Find the T-annihilator of the vector $\alpha = (2, -2, 0)$.
(c) Find a cyclic vector for T.

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(2) Provide a 7×7 complex matrix A in rational form such that:

- A has one and only one distinct characteristic value
- Min. Poly. (A) = $(x - 1)^3$
- A has five invariant factors.

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(3) Give all possible Jordan forms for a linear operator T with minimal polynomial $x^2(x - 1)^2$ and characteristic polynomial $x^4(x - 1)^4$, and such that the kernel of T has dimension 2.

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(4) Let V be an inner product space and W a finite-dimensional subspace of V. Prove that if E is a projection (i.e., $E^2 = E$), such that $\|E(\alpha)\| \le \|\alpha\|$ $\forall$ $\alpha \in V$, then E is the orthogonal projection of V on W := range(E).

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(5) Let V be a finite-dimensional inner product space.
(a) Prove that any orthogonal projection of V (on a subspace) is self-adjoint.
(b) Let E be a projection (i.e., $E^2 = E$). Prove that if E is normal, then E and E* have the same nullspace; and then show that $V = W \oplus W^{\perp}$, where W := range(E).
(c) Use (a) & (b) to show that a projection is normal if and only if it is self-adjoint.

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(6) Let q be the quadratic form on R^2 given by $q(x, y) = 2bxy$, where $0 \neq b \in R$. Find an invertible linear operator U on R^2 such that $q(U(x, y)) = 2bx^2 - 2by^2$.

---------------------------- *good luck* ------------------------------