King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

Math 550 – Linear Algebra (Term 161)

Midterm Exam

Notes:

- Duration = **3 hours.**
- Each problem is worth **10 points.**

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(1) Let F be a field and let n be a positive integer ($n \geq 2$). Let V be the vector space of all $n \times n$ matrices over F. Which of the following subsets of V are subspaces of V?

(a) $E_1 = \{A \in V \mid A \text{ invertible}\}$;

(b) $E_2 = \{A \in V \mid A \text{ non-invertible}\}$;

(c) $E_3 = \{A \in V \mid AB = BA\}$; where B is some fixed matrix in V

(d) $E_4 = \{A \in V \mid A^2 = A\}$.

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(2) Let V be the vector space over the complex numbers of all functions from R into C, i.e., the space of all complex-valued functions on the real line.

Let $f_1 = 1$, $f_2 = e^{ix}$, $f_3 = e^{-ix}$ and let $g_1 = 1$, $g_2 = \cos(x)$, $g_3 = \sin(x)$.

(a) Prove $B = \{f_1, f_2, f_3\}$ is linearly independent.

(b) Find an invertible matrix $P = (p_{ij})$ such that $g_j = \sum_{1 \leq i \leq 3} p_{ij} f_i$, for $j = 1, \ldots, 3$

(c) Compute P^{-1}.

(d) Is B a basis for V?

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(3) Let n be a positive integer and let V be an n-dimensional vector space over a field F. Let $A = (a_{ij})$ be an $n \times n$ matrix given by $a_{ij} = 1$ for $i = j + 1$, $1 \leq j \leq n - 1$; and $a_{ij} = 0$, otherwise.

(a) Let T be a linear operator on V such that $T^n = 0$ and $T^{n-1} \neq 0$. Prove that there is an ordered basis B for V such that $[T]_B = A$.

(b) Use (a) to prove the following result: Let M and N be two $n \times n$ matrices such that $M^n = N^n = 0$, $M^{n-1} \neq 0$, and $N^{n-1} \neq 0$. Then M and N are similar.

(4) Let n be a positive integer and F a field. Consider the two subspaces, respectively, of F^n and $(F^n)^*$ given by:

$$W = \{(x_1, \ldots, x_n) \in F^n \mid \sum_{1 \le i \le n} x_i = 0\} \text{ and}$$

$$W_1 = \{f \in (F^n)^* \mid f(x_1, \ldots, x_n) = \sum_{1 \le i \le n} c_i x_i \text{ with } \sum_{1 \le i \le n} c_i = 0 \text{ , } c_i \in F\}.$$

Define the linear transformation $\varphi : W_1 \to W^*$, $f \to \varphi(f)$, where $\varphi(f) : W \to F$ denotes the restriction of f to W.

(a) Prove that $W^o = \{f \in (F^n)^* \mid f(x_1, \ldots, x_n) = c\sum_{1 \le i \le n} x_i \text{ , } c \in F\}$.

(b) Prove that φ is nonsingular.

(c) Deduce that W_1 and W^* can be naturally identified.

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(5) Let a, b, and c be elements of a field F and let A be the following matrix over F:

$$\begin{pmatrix} 0 & 0 & c \\ 1 & 0 & b \\ 0 & 1 & a \end{pmatrix}$$

Let f and p denote the characteristic polynomial and minimal polynomial of A, respectively.

(a) Find f.

(b) Find p.

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(6) Let a be a nonzero real number and consider the following real matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}.$$

Let T_1 and T_2 be linear operators on V = $\mathbb{R}^2$ associated, respectively, to A and B in the standard basis. Let p_1, p_2 denote the minimal polynomials of T_1 and T_2, respectively.

(a) Find p_1 and p_2.

(b) Prove the existence of an ordered basis B for V such that $[T_1]_B$ and $[T_2]_B$ are both diagonal. [Do not construct B at this stage]

(c) Construct B explicitly; and use it to find an invertible real matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are both diagonal.

------------------------------------ *Good Luck* --