King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Semester 161 - October 29, 2016

Math 513 Exam I

Name: ______________________________ ID# ____________________

[Q1: 13 points, Q2: 11 points, Q3: 4 points, Q4: 9 points, Q5: 5 points, Q6: 8 points]

1. Consider $f(\mathrm{t}) = \begin{cases} -t-\pi & if \quad -\pi \le t < 0 \\ 0 & if \quad 0 \le t < \pi \end{cases}$.
   a) Find the Fourier series of $f(t)$ on the given interval.
   b) What does the series converge to when $t = -\pi/2,\ 0, 2\pi$?
   c) Using the series obtained in (a), evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$.
2. Consider the function $f(\mathrm{t}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nt)$
   a) Write $f(\mathrm{t})$ in the cosine phase angle form.
   b) Solve the differential equation $y'' + y = f(\mathrm{t})$.
3. Sketch and then find the Fourier transform of $H(t+\pi) - H(t-\pi)$, where H is the Heaviside function. Write your answer in trigonometric functions and fully simplify it.
4.
   a) Given that the Fourier transform of $e^{-a|t|}$, $a > 0$ is $\frac{2a}{\omega^2 + a^2}$, use the symmetry (Duality-time frequency) property to find the Fourier transform of $\frac{1}{1+9t^2}$.
   b) Use Parseval's equality to show that $\int_0^{\infty} \frac{d\omega}{\left(\omega^2 + a^2\right)^2} = \frac{\pi}{4a^3}$.
5. Find the inverse Fourier transform of $\frac{e^{-2\omega i}}{3 + (\omega - 5)i}$.
6.
   a) Use partial fractions to find the inverse Fourier transform of $\frac{1}{(i\omega+1)\,(2i\omega+1)}$.
   b) Use Fourier transform to find a particular solution of $2y'' + 3y' + y = \delta(t-1)$, where δ is the Dirac delta function.

Useful Formulas

$$\int_a^b t\cos[nt]\,\mathrm{d}t = \frac{-\cos[an]+\cos[bn]-an\sin[an]+bn\sin[bn]}{n^2}$$

$$\int_a^b t\sin[nt]\,\mathrm{d}t = \frac{an\cos[an]-bn\cos[bn]-\sin[an]+\sin[bn]}{n^2}$$

Fourier Transform

$$\mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt = F(\omega)$$

$$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t} d\omega = f(t)$$

$$\mathcal{F}\{t^n\, H(t)\, e^{-at}\} = n!/(a+i\,\omega)^{n+1}, \mathrm{Re}(a) > 0, n = 1,2,\ldots$$

$$\mathcal{F}\{\mathrm{sgn}(t)\} = \begin{cases} 2/(i\omega), & \omega \neq 0 \\ 0, & \omega = 0 \end{cases}$$

$$\mathcal{F}\{f(t-\tau)\} = e^{-i\omega\tau}F(\omega)$$

$$\mathcal{F}\{f(t)e^{i\omega_0 t}\} = F(\omega-\omega_0)$$

$$\mathcal{F}\{f^n(x)\} = (i\omega)^n F(\omega)$$

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(x)\, g(t-x)\, dx$$

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$