King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 355: Linear Algebra

Final Exam, Fall Semester 161 (180 minutes)

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Q1. (30 points) Prove the following:

(a) If V is a vector space and $A : V \to V$ is a linear map, then any set of eigenvectors of A with *distinct* eigenvalues is linearly independent.

(b) If V is a non-zero finite dimensional complex vector space, then every linear operator $A : V \to V$ has a *fan* in V.

(c) If A is an $n \times n$ matrix over a field K, then

$$I := \{f \in K[x] \mid f(A) = 0\} \neq \{0\}.$$

Q2. (20 points) Let

$$A = \begin{bmatrix} 7 & 0 & -10 \\ 5 & 2 & -10 \\ 5 & 0 & -8 \end{bmatrix}$$

(a) Find the *characteristic polynomial* of A.
(b) Find the *minimal polynomial* of A.
(c) Find the eigenvalue(s) and the corresponding eigenspace(s) of A.
(d) Is A diagonalizable? (Explain)

Q3. (10 points) Consider the subspace

$$W := Span\left(\left\{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -3 \end{pmatrix}\right\}\right)$$

of $\mathbb{R}^4$. Find $\dim(W^{\perp})$.

Q4. (10 points) Let V be a non-zero finite dimensional *complex* vector space. Show that any linear operator $A : V \to V$ can be written as a sum of two linear operators $A = D + N$, where D is diagonalizable and N is nilpotent (*i.e.* $N^m = 0$ for some positive integer m).

Q5. (30 points) Prove or disprove:

(a) Every symmetric *real* matrix is diagonalizable.

(b) Every linear operator on a *complex* vector space has an eigenvalue.

(c) Any map from $A : \mathbb{R} \to \mathbb{R}$ (considered as a vector space over $\mathbb{Q}$) with the property that $A(qv) = qA(v)$ for all $q \in \mathbb{Q}$ and $v \in \mathbb{R}$ is *linear*.

GOOD LUCK