King Fahd University of Petroleum & Minerals

Department of Mathematics and Statistics

Math 355: Linear Algebra

First Exam, Fall Semester 161 (120 minutes)

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Q1. (10 points) Let V and W be vector spaces over the field K, with $\dim(V) < \infty$, and $L : V \longrightarrow W$ be a linear map. Show that

$$\dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = \dim(V).$$

Q2. (20 points) (a) Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an invertible linear map. Show that if A is the matrix associated with F, then A^{-1} is the matrix associated with the inverse of F.

(b) Let $F_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be rotation by an angle θ counterclockwise. Show that F_θ is invertible and determine the matrix associated with F_θ^{-1}. Show that $\|F_\theta(X)\| = \|X\|$ for all $X \in \mathbb{R}^2$ (where $\|(a,b)\| := \sqrt{a^2+b^2}$).

Q3. (20 points) Consider the linear map

$$L : \mathbb{R}^4 \longrightarrow \mathbb{R}^2, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \mapsto \begin{pmatrix} x_1 - x_2 - x_3 \\ x_2 + x_3 - x_4 \end{pmatrix}$$

(a) Find $\mathrm{Ker}(L)$, a basis for $\mathrm{Ker}(L)$ and $\dim(\mathrm{Ker}(L))$.
(b) Find $\mathrm{Im}(L)$, a basis for $\mathrm{Im}(L)$ and $\dim(\mathrm{Im}(L))$.

Q4 (20 points) Consider the linear map

$$tr : M_{n\times n}(K) \longrightarrow K, \ (a_{ij}) \mapsto \sum_{i=1}^{n} a_{ii}.$$

(a) Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A, B \in M_{n\times n}(K)$.
(b) Find the dimension of the subspace

$$W := \{A \in M_{n\times n}(K) \mid \mathrm{tr}(A) = 0\}.$$

Q5. (30 points) Prove or disprove:
(a) There are *no* $n \times n$ matrices A and B such that $AB - BA = I_n$.
(b) The zero vector space is one dimensional.
(c) For any field K, any matrix $A \in M_{n\times n}(K)$ cab be written as $A = B + C$, where ${}^tB = B$ and ${}^tC = -C$.

GOOD LUCK