King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

MATH 302, Semester 161 (2016-2017)

FINAL EXAM

January 12, 2017

Allowed Time: 3 Hours

Student Name:

Student ID Number:

Section Number:

Instructor's Name:

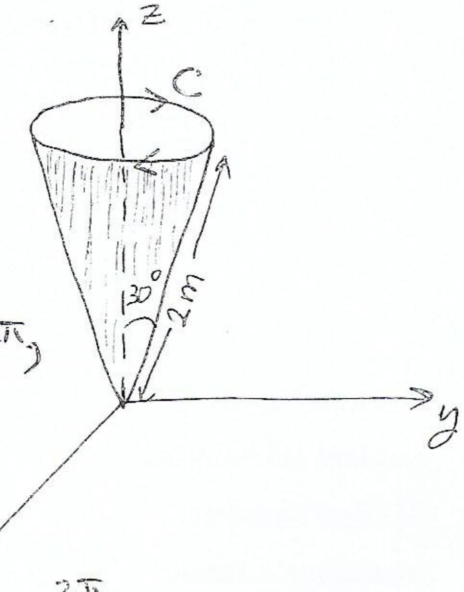
Instructions:

1. Write neatly and legibly -- *you may lose points for messy work.*
2. Show all your work -- *no points for answers without justification.*
3. Calculators and mobiles are not allowed.
4. Make sure that you have 11 pages (8 questions+1 cover page+2 formula sheet).

Problem No.	Points	Maximum Points
1		20
2		10
3		30
4		15
5		20
6		15
7		15
8		15
Total:		**140**

Coordinator: Dr A. N. Duman

Q1. If $E = r\,\hat{a}_r + \cos\phi\,\hat{a}_\theta - r^2\cos\theta\,\hat{a}_\varphi$, verify Stokes's theorem for the open surface shown in the figure.



Sol.

$$\oint_C E\cdot dl = \int_S (\nabla\times E)\cdot ds$$

on the curve C: $r=2,\ \theta=\frac{\pi}{6},\ 0\le\varphi\le 2\pi,$

$$dl = r\sin\theta\, d\varphi\, \vec{a}_\varphi$$

so, $$\oint_C E\cdot dl = -\int_0^{2\pi} -r^3\cos\theta\sin\theta\, d\varphi = 8\cos\frac{\pi}{6}\sin\frac{\pi}{6}\int_0^{2\pi} d\varphi = 4\sqrt{3}\,\pi.$$

Next, $$\nabla\times E = \frac{1}{r^2\sin\theta}\begin{vmatrix} \vec{a}_r & r\vec{a}_\theta & r\sin\theta\,\vec{a}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} \\ r & r\cos\varphi & -r^3\cos\theta\sin\theta \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[-r^3\cos 2\theta + r\sin\varphi\right]\vec{a}_r - \frac{r\vec{a}_\theta}{r^2\sin\theta}\left[-3r^2\cos\theta\sin\theta\right]$$

$$+ \frac{1}{r^2\sin\theta}\left[\cos\varphi\right] r\sin\theta\,\vec{a}_\varphi$$

$$= \frac{1}{r\sin\theta}\left[-r^2\cos 2\theta + \sin\varphi\right] + 3r\cos\theta\,\vec{a}_\theta + \frac{\cos\varphi}{r}\vec{a}_\varphi.$$

On S: $\theta=\frac{\pi}{6},\ 0\le\varphi\le 2\pi,\ 0\le r\le 2\pi,$

$$ds = r\sin\theta\, dr\, d\varphi\, \vec{a}_\theta.$$

$$\int(\nabla\times E)\cdot ds = \int_0^{2\pi}\int_0^{2} 3r^2\sin\theta\cos\theta\, dr\, d\varphi = 2\pi\frac{\sqrt{3}}{2}\cdot\frac{1}{2}\, r^3\Big]_0^2$$

Q2. Find all points $z = x + iy$ for which $f(z) = \cos x \cosh y - i \sin x \sinh y$ is differentiable. Compute $f'(z)$ whenever it exists.

Sol. $f(z) = u(x,y) + i\,v(x,y)$

$$\frac{\partial u}{\partial x} = -\sin x \cosh y, \qquad \frac{\partial u}{\partial y} = \cos x \sinh y$$

$$\frac{\partial v}{\partial x} = -\cos x \sinh y, \qquad \frac{\partial v}{\partial y} = -\sin x \cosh y$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, for all x, y

and u_x, u_y, v_x, v_y are continuous, for all x, y

then f is differentiable for all z.

$$f'(z) = u_x + i\,v_x = -\sin x \cosh y - i \cos x \sinh y$$

Q3.

(a) Solve the equation $z^{10} - 2z^5 + 4 = 0$.

(b) Solve the equation $(\sin z + \cos z)^2 = 3$

Sol. (a) let $W = z^5$ then, we have

$$W^2 - 2W + 4 = 0 \Rightarrow W = \frac{2 \pm i\sqrt{12}}{2} = 1 \pm i\sqrt{3}$$

That is, $W = z^5 = 2e^{i(\pi/3 + 2k\pi)}$ or $2e^{i(-\frac{\pi}{3} + 2k\pi)}$

Therefore

$$z = 2^{1/5} e^{i\left(\frac{\pi}{15} + \frac{2k\pi}{5}\right)}, \quad k = 0, 1, 2, 3, 4.$$

$$z = 2^{1/5} e^{i\left(-\frac{\pi}{15} + 2k\pi\right)}, \quad k = 0, 1, 2, 3, 4.$$

(b) $(\sin z + \cos z)^2 = 3 \Leftrightarrow \cos^2 z + \sin^2 z + 2\sin z\cos z = 3$

$\Leftrightarrow 2\sin z\cos z = 2 \Leftrightarrow \boxed{\sin 2z = 2}$

$$\frac{e^{2iz} - e^{-2iz}}{2i} = 2 \Leftrightarrow e^{2iz} - e^{-2iz} = 4i$$

let $W = e^{2iz}$ then $W - \frac{1}{W} = 4i$

$\Rightarrow W^2 - 4iW - 1 = 0$

$\Rightarrow W = \frac{4i \pm i\,2\sqrt{3}}{2} = (2 \pm \sqrt{3})\,i$

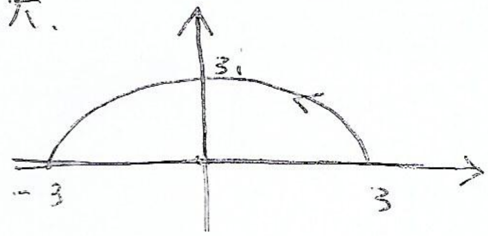
$\Rightarrow 2iz = \log(2 \pm \sqrt{3})i$

$= \ln(2 \pm \sqrt{3}) + i(\pi/2 + 2k\pi)$

$\Rightarrow \boxed{z = \frac{\pi}{4} + k\pi - \frac{i}{2}\ln(2 \pm \sqrt{3})}$ k integer

Q4. Evaluate $\int_C z\,Re(z^2)dz$ along the contour C, which is the positively oriented upper half circle centered at the origin with a radius 3.

C is given by $z = 3e^{it}$, $0 \le t \le \pi$.



So

$$Re(z^2) = Re(9e^{2it}) = 9\cos 2t$$

$$\int_C z\,Re(z^2)\,dz = \int_0^\pi (3e^{it})(9\cos 2t)\,3i\,e^{it}\,dt$$

$$= 81i\int_0^\pi e^{2it}\cos 2t\,dt$$

$$= 81i\int_0^\pi (\cos 2t + i\sin 2t)\cos 2t\,dt$$

$$= 81i\int_0^\pi \cos^2 2t\,dt - 81\int_0^\pi \sin 2t\cos 2t\,dt$$

$$= \frac{81i}{2}\int_0^\pi (1+\cos 4t)\,dt \cancel{- \frac{81}{2}\left[\sin^2 2t\right]_0^\pi}$$

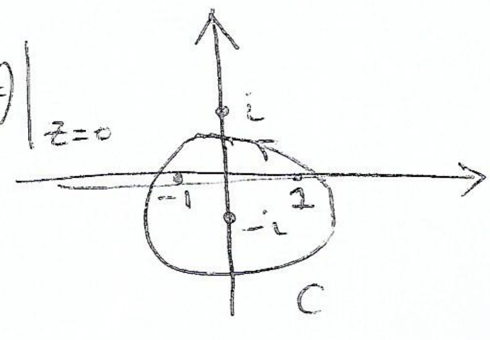
$$= \frac{81i}{2}\pi \cancel{-\left[\frac{81i}{8}\sin 4t\right]_0^\pi}$$

$$= \frac{81i\pi}{2}.$$

Q5. Let C be the circle $|z+i| = \frac{3}{2}$. Use Cauchy's integral formula to find $g(0)+g(1)$, if

$$g(a) = \oint_C \left[\frac{\sin z}{(z^2+a)^2}\right] dz.$$

$$g(0) = \oint_C \frac{\sin z \, dz}{z^4} = \frac{2\pi i}{3!} \frac{d^3}{dz^3}(\sin z)\Big|_{z=0}$$



$$= \frac{\pi i}{3}(-\cos 0) = \frac{-\pi i}{3}.$$

$$g(1) = \oint_C \frac{\sin z}{(z^2+1)^2} dz = \oint_C \frac{\sin z}{(z-i)^2(z+i)^2}$$

$$= \oint_C \frac{\frac{\sin z}{(z-i)^2}}{(z+i)^2} dz = \frac{2\pi i}{1!} \frac{d}{dz}\left(\frac{\sin z}{(z-i)^2}\right)\Big|_{z=-i}$$

$$= 2\pi i \, \frac{(z-i)^2 \cos z - 2(z-i)\sin z}{(z-i)^4}\Big|_{z=-i}$$

$$= 2\pi i \, \frac{-4\cos i + 4i \sin(-i)}{16}$$

$$= -\frac{\pi i}{2}(\cos i + \sin i).$$

$$g(0)+g(1) = -\frac{\pi i}{3} - \frac{\pi i}{2}(\cos i + \sin i)$$

Q6. Give the Laurent series for the function

$$f(z) = \frac{z^2+z-3}{z^2-z-2}$$

in the domain $1 < |z| < 2$.

Sol.

$$f(z) = 1 + \frac{2z-1}{z^2-z-2}$$

$$\frac{2z-1}{z^2-z-2} = \frac{2z-1}{(z-2)(z+1)} = \frac{a}{z-2} + \frac{b}{z+1} =$$

$$= \frac{1}{z-2} + \frac{1}{z+1}$$

$$\frac{1}{z-2} = -\frac{1}{2}\,\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k = \sum_{k=0}^{\infty} -\frac{z^k}{2^{k+1}},$$

$$|z| < 2.$$

$$\frac{1}{1+z} = \frac{1}{z}\,\frac{1}{1+\frac{1}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}, \quad |z| > 1.$$

Thus

$$f(z) = 1 - \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \cdots + \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots$$

$$= \cdots\cdots + \frac{1}{z^3} - \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \cdots\cdots$$

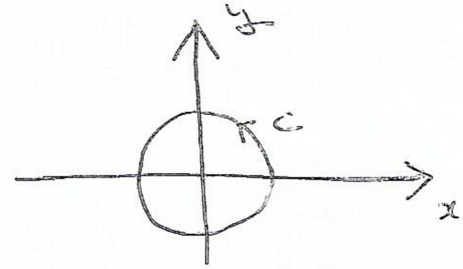
$$1 < |z| < 2$$

Q7. Evaluate the integral,

$$\oint_C \left(z^4 \sin\left(\frac{1}{2z}\right) + \frac{e^z}{z^2 - 4i} \right) dz$$

along the contour defined by $|z| = 1$.

$$f(z) = z^2 \sin \frac{1}{2z} + \frac{e^z}{z^2 - 4i}$$



has singularities at $z = 0$ and at $z^2 - 4i = 0$. That is $|z|^2 = |4i| = 4 \Leftrightarrow |z| = 2$

So the zeros of $z^2 - 4i = 0$ are outside C

hence $\oint_C \frac{e^z}{4z^2 - 4i} dz = 0$ by Cauchy.

$$z^4 \sin \frac{1}{2z} = z^4 \left[\frac{1}{2z} - \frac{1}{3!} \frac{1}{(2z)^3} + \frac{1}{5!} \frac{1}{(2z)^5} - \cdots \right]$$

$$= \frac{z^4}{2} - \frac{z}{48} + \frac{1}{5! \cdot 2^5} \frac{1}{z} - \cdots$$

$$\Rightarrow \quad \text{Res}(f, 0) = \frac{1}{5!\, 2^5}$$

$$\Rightarrow \oint_C \left(z^2 \sin \frac{1}{2z} + \frac{e^z}{z^2 - 4i} \right) dz = \frac{2\pi i}{5!\, 2^5} = \frac{\pi i}{5!\, 2^4}$$

Q8. Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4+5x^2+4}.$$

Sol. $z^4+5z^2+4 = (z^2+1)(z^2+4)$

So the denominator $\varphi(z)$ has four zeros $\pm i, \pm 2i$

Only i and $2i$ are in the upper half plane

and they are simple poles for

$$f(z) = \frac{z^2}{(z+i)(z-i)(z+2i)(z-2i)}.$$

$$\mathrm{Res}(f,i) = \lim_{z\to i}(z-i)f(z) = \frac{-1}{(2i)(3i)(-i)} = \frac{-1}{6i}$$

$$\mathrm{Res}(f,2i) = \lim_{z\to 2i}(z-2i)f(z) = \frac{-\cancel{4}}{(3i)(i)(\cancel{4}i)} = \frac{1}{3i}$$

So

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i\left[\frac{1}{3i} - \frac{1}{6i}\right] = \frac{2\pi}{6} = \frac{\pi}{3}.$$