King Fahd University of Petroleum & Minerals
Department of Mathematics and Statistics

MATH 302, Semester 161 (2016-2017)

EXAM I

October 26, 2016

Allowed Time: 2 Hours

Student Name:
Student ID Number: Solution
Section Number:
Serial Number:
Instructor's Name:

Instructions:

1. Write neatly and legibly -- *you may lose points for messy work.*
2. Show all your work -- *no points for answers without justification.*
3. Programmable Calculators and Mobiles are not allowed.
4. Make sure that you have 6 different problems (6 pages + cover page).

Problem No.	Points	Maximum Points
1		15
2		15
3		20
4		20
5		15
6		10
Total:		100

Coordinator: Dr Ali N Duman

Q1. (a) Determine whether, $S = \{\langle x, y, z, t\rangle \mid xy = zt\}$ is a subspace of $\mathbb{R}^4$? ***[5 points]***

(b) Show that $V = \{\langle x, y, z, w\rangle \mid x + 2y = z + 3w = 0\}$ is a subspace of $\mathbb{R}^4$. ***[5 points]***

(c) Find a basis and for the subspace, $V = \{\langle x, y, z, w\rangle \mid x + 2y = z + 3w = 0\}$ of $\mathbb{R}^4$. What is the dimension of this subspace? ***[5 points]***

Sol.

(a) $a = \langle 1, 1, 1, 1\rangle \in S$, $b = \langle 2, 2, 4, 1\rangle \in S$

$a + b = \langle 3, 3, 5, 2\rangle$. But $(3)(3) \neq (5)(2)$.

So S is not a subspace.

(b) • If $a \in V$ then $a = \langle -2y, y, -3w, w\rangle$

• clearly, if $y = w = 0$ then $0 \in V$

• let $a = \langle -2y, y, \cancel{-}3w, w\rangle \in V$, $b = \langle -2s, s, -3t, t\rangle \in V$

then $a + b = \langle -2(y+s), y+s, -3(w+t), w+t\rangle \in V$

• $ka = \langle -2ky, ky, -3kw, kw\rangle \in V$

So V is a subspace of $\mathbb{R}^4$.

(c) Basis and dimension

$a \in V \iff a = \langle -2y, y, -3w, w\rangle$

$= y\langle -2, 1, 0, 0\rangle + w\langle 0, 0, -3, 1\rangle$

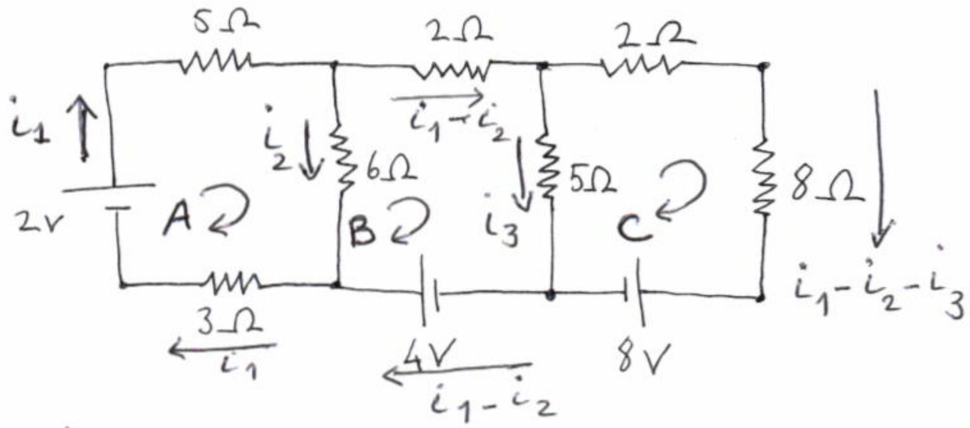
Thus $B = \{\langle -2, 1, 0, 0\rangle, \langle 0, 0, -3, 1\rangle\}$ is a basis for V.

$\dim V = 2$.

Q2. Using **Gaussian elimination**, find the currents in all branches of the circuit below.

[15 points]

Hint: First, reduce the number of variables to three using Kirchoff's point rule.



in A: $-5i_1 - 6i_2 - 3i_1 + 2 = 0$

in B: $-2(i_1 - i_2) - 5i_3 + 6i_2 + 4 = 0$

in C: $10(i_1 - i_2 - i_3) - 5i_3 + 8 = 0$

We then obtain

$$8i_1 + 6i_2 = 2 \iff 4i_1 + 6i_2 = 1$$

$$2i_1 - 8i_2 + 5i_3 = 4$$

$$-10i_1 + 10i_2 + 15i_3 = 8$$

Gauss-Jordan

$$\left(\begin{array}{ccc|c} 8 & 6 & 0 & 2 \\ 2 & -8 & 5 & 4 \\ -10 & 10 & 15 & 8 \end{array}\right) \xrightarrow{R_1 \leftrightarrow \frac{R_2}{2}} \left(\begin{array}{ccc|c} 1 & -4 & 5/2 & 2 \\ 8 & 6 & 0 & 2 \\ -10 & 10 & 15 & 8 \end{array}\right) \xrightarrow[R_3 + 10R_1]{R_2 - 8R_1}$$

$$\left(\begin{array}{ccc|c} 1 & -4 & 5/2 & 2 \\ 0 & 38 & -20 & -14 \\ 0 & -30 & 40 & 28 \end{array}\right) \xrightarrow[R_3/2]{R_2/38} \left(\begin{array}{ccc|c} 1 & -4 & 5/2 & 2 \\ 0 & 1 & -\frac{10}{19} & -7/19 \\ 0 & -15 & 20 & 14 \end{array}\right) \xrightarrow[R_3 + 15R_2]{R_1 + 4R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{15}{38} & \frac{10}{19} \\ 0 & 1 & -10/19 & -7/19 \\ 0 & 0 & \frac{230}{19} & \frac{161}{19} \end{array}\right) \Longrightarrow \begin{array}{l} i_3 = \frac{161}{230} = \frac{7}{10} \\ i_2 = -\frac{7}{19} + \frac{\cancel{10}}{19} \cdot \frac{7}{\cancel{10}} = 0 \Rightarrow i_1 = \frac{1}{4}. \end{array}$$

The solution has 3 parameters.

Since the system is consistent, then

rank of $\tilde{B}$ = rank $B = 2$.

Q3. (a) Consider the system of non-homogenous linear algebraic equations,

$$\begin{aligned} ax_1 \qquad\quad &+ x_3 = 161 \\ -x_1 \quad +ax_2 \qquad &\quad\;\; = 302 \\ x_2 \quad &+ x_3 = 2016 \end{aligned}$$

If there is no parameters in the solution of the consistent system, what are the values that a can **NOT** have? *[10 points]*

(b) Consider the matrix

$$B = \begin{pmatrix} 3 & 6 & -1 & -5 & 5 \\ 2 & 4 & -1 & -3 & 2 \\ 3 & 6 & -2 & -4 & 1 \end{pmatrix}$$

If the system $BX = C$ is consistent. How many parameters does the solution have? What is the rank of the augmented matrix $(B|C)$?

[10 points]

Sol. The system has only solution and A is of full rank. That is rank $A = 3$

$$\begin{pmatrix} a & 0 & 1 \\ -1 & a & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow -R_1} \begin{pmatrix} 1 & -a & 0 \\ a & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_2 - aR_1} \begin{pmatrix} 1 & -a & 0 \\ 0 & a^2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & -a & 0 \\ 0 & 1 & 1 \\ 0 & a^2 & 1 \end{pmatrix} \xrightarrow[R_3 - a^2 R_2]{R_1 + aR_2} \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 1 \\ 0 & 0 & 1-a^2 \end{pmatrix}$$

So, rank $A = 3$ iff $a \neq \pm 1$.

(b) $$\begin{pmatrix} 3 & 6 & -1 & -5 & 5 \\ 2 & 4 & -1 & -3 & 2 \\ 3 & 6 & -2 & -4 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1} \begin{pmatrix} 3 & 6 & -1 & -5 & 5 \\ 2 & 4 & -1 & -3 & 2 \\ 0 & 0 & -1 & 1 & -4 \end{pmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow \frac{R_2}{2}} \begin{pmatrix} 1 & 2 & -\frac{1}{2} & -\frac{3}{2} & 1 \\ 3 & 6 & -1 & -5 & 5 \\ 0 & 0 & -1 & 1 & -4 \end{pmatrix} \xrightarrow{R_2 - 3R_1} \begin{pmatrix} 1 & 2 & -\frac{1}{2} & -\frac{3}{2} & 1 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 2 \\ 0 & 0 & -1 & 1 & -4 \end{pmatrix}$$

$$\xrightarrow[R_3 + 2R_2]{R_1 + R_2} \begin{pmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{2R_2} \begin{pmatrix} 1 & 2 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Q4. (a) Find the eigenvalues of the matrix,

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{pmatrix}$$

[10 points]

(b) Find a vector $\boldsymbol{v}$ such that $A^{302}\boldsymbol{v} = \boldsymbol{v}$. *[10 points]*

Sol.

(a) $$\begin{vmatrix} 3-\lambda & 0 & 0 \\ -4 & 6-\lambda & 2 \\ 16 & -15 & -5-\lambda \end{vmatrix} = (3-\lambda)\left[(6-\lambda)(-5-\lambda)+30\right]$$

$$= (3-\lambda)(\lambda^2-\lambda-30+30) =$$

$$= \lambda(3-\lambda)(\lambda-1) = 0$$

Eigenvalues are $\lambda = 0, 1, 3$.

(b) For $\lambda = 1$, the eigenvector v satisfies

$Av = v \Rightarrow A^2 v = Av = v \Rightarrow \dots\ A^{302} v = v$

So we find v:

$$\begin{pmatrix} 2 & 0 & 0 \\ -4 & 5 & 2 \\ 16 & -15 & -6 \end{pmatrix} \xrightarrow[R_2+2R_1]{R_3-8R_1} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & -15 & -6 \end{pmatrix} \xrightarrow[R_3+3R_2]{R_1/2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\cancel{So}$ $v_1 = 0 \quad 5v_2 + 3v_3 = 0 \Rightarrow v_2 = -\dfrac{3v_3}{5}$

If $v_3 = 5$ then $v_2 = -3$

$$\therefore\ v = \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix}.$$

$$\boxed{\lambda = 2} \quad \begin{pmatrix} -2 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ -2 & -1 & +1 \end{pmatrix} \xrightarrow[R_3 + 2R_1]{R_2 + R_1}$$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow[R_3 - R_2]{-R_2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\Rightarrow \quad x_1 = x_3, \quad x_2 = -x_3$$

an eigenvector is $E_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$$P = \begin{pmatrix} \frac{-2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

(b) Eigenvalues of A^{-1}

$$\lambda_1 = -1, \quad E_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1, \quad E_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_3 = \frac{1}{2}, \quad E_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Q5. (a) Consider

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find matrices P and D, with P orthogonal and D diagonal, such that $A = PDP^{-1}$. *[10 points]*

(b) Find the eigenvalues and eigenvectors of A^{-1}. *[5 points]*

(a) $\begin{vmatrix} -\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1-\lambda & 0 \end{vmatrix} + (1-\lambda)\begin{vmatrix} -\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix}$

$= (\lambda - 1) + (1-\lambda)[\lambda^2 - \lambda - 1]$

$= -(\lambda - 1)[-1 + \lambda^2 - \lambda - 1] = (1-\lambda)(\lambda+1)(\lambda-2) = 0$

$\Rightarrow$ Eigenvalues are $\lambda = \pm 1, 2$

Eigenvectors

$\boxed{\lambda = -1}$ $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 + R_1} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - R_2]{R_1 + R_2}$

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. So $\begin{array}{l} x_1 = -2x_3 \\ x_2 = -x_3 \end{array}$

an eigenvector is $E_1 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$.

$\boxed{\lambda = 1}$ $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{R_2 - R_1 \\ R_3 + R_1}]{-R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow[R_3 + R_2]{R_1 - R_2}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 = 0, \quad x_2 = x_3$

an eigenvector is $E_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

Q6. Find the inverse of the matrix A, if it exists, using the augmented matrix method where

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

Show the step by step calculations for full marks. *[10 pts]*

Sol.

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 - 3R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 6 & -5 & -3 & 1 & 0 \end{array}\right) \xrightarrow[R_3 - 6R_2]{R_1 + 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -5 & +3 & 1 & -6 \end{array}\right) \xrightarrow{R_3/-5}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & +\frac{6}{5} \end{array}\right) \xrightarrow{R_1 - 2R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{3}{5} & -\frac{1}{5} & +\frac{6}{5} \end{array}\right)$$

So $$A^{-1} = \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & -\frac{2}{5} \\ -1 & 0 & 1 \\ -\frac{3}{5} & -\frac{1}{5} & \frac{6}{5} \end{pmatrix}$$