Q1. Let $A = 3z^2 \sin\varphi\, \hat{a}_\rho + \rho\cos 2\varphi\, \hat{a}_\varphi - \rho z \hat{a}_\varphi$ at . $P(2\sqrt{3}, \frac{\pi}{6}, 2)$

(a) Determine the vector component of A that is tangential to the surface $\theta = \pi/3$. *[15pts]*

(b) Determine the angle that A makes the tangent plane of the surface $r = 4$. *[10 pts]*

Solution (a) At the point P, $A = 6\hat{a}_\rho + \sqrt{3}\,\hat{a}_\varphi - 4\sqrt{3}\,\hat{a}_z$ 2

The surface given by $G(r,\theta,\varphi) = \theta = \pi/3$. 2

So the normal to the surface is $\nabla G = \langle 0,1,0\rangle = \hat{a}_\theta$.

Normal in the Cartesian coordinates:

$$\begin{pmatrix} N_x \\ N_y \\ N_z \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\theta\cos\varphi & -\sin\varphi \\ \sin\theta\sin\varphi & \cos\theta\sin\varphi & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

3.5

$$= \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} \sqrt{3}/4 \\ 1/4 \\ -\sqrt{3}/2 \end{pmatrix}, \quad \left(\theta = \frac{\pi}{3},\ \varphi = \frac{\pi}{6}\right)$$

1

Normal in cylindrical

$$\begin{pmatrix} N_\rho \\ N_\varphi \\ N_z \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3}/4 \\ 1/4 \\ -\sqrt{3}/2 \end{pmatrix}$$

3.5

$$= \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{3}/4 \\ 1/4 \\ -\sqrt{3}/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 0 \\ -\sqrt{3}/2 \end{pmatrix} = \frac{1}{2}\hat{a}_\rho - \frac{\sqrt{3}}{2}a_z$$

Component of A parallel to normal is

2 $$A_N = \frac{A\cdot N}{\|N\|^2} N = \frac{3 - 0 + 6}{1} N = 9N = \frac{9}{2}\hat{a}_\rho - \frac{9\sqrt{3}}{2}\hat{a}_z$$

Tangent component is $A_T = A - A_N$

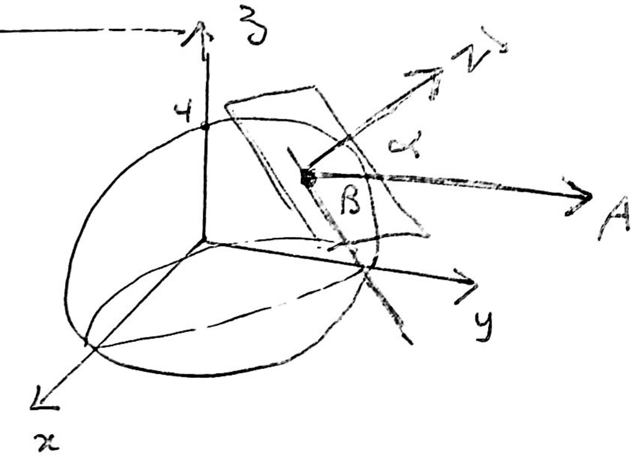
1

$$A_T = \frac{3}{2}\hat{a}_\rho + \sqrt{3}\,\hat{a}_\varphi + \frac{\sqrt{3}}{2}\hat{a}_z$$

(b) The normal to the surface $r = 4 = \psi(r,\theta,\varphi)$

is $\nabla\psi = \langle 1,0,0\rangle = \vec{a}_r$ (spherical) 2

Normal in Cartesian



$$N = \begin{pmatrix} \sin\frac{\pi}{3}\cos\frac{\pi}{6} & \cos\frac{\pi}{3}\cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{3}\sin\frac{\pi}{6} & \cos\frac{\pi}{3}\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \\ \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} & 0 \end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}3/4\\ \sqrt{3}/4 \\ 1/2\end{pmatrix}$$

$$= \frac{3}{4}\hat{a}_x + \frac{\sqrt{3}}{4}\hat{a}_y + \frac{1}{2}\hat{a}_z \quad 3$$

Normal in cylindrical

$$\begin{pmatrix} N_\rho \\ N_\varphi \\ N_z \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & \frac{1}{2} & 0 \\ -\frac{1}{2} & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix}3/4 \\ \sqrt{3}/4 \\ 1/2\end{pmatrix} = \begin{pmatrix}\sqrt{3}/2 \\ 0 \\ 1/2\end{pmatrix} = \frac{\sqrt{3}}{2}\hat{a}_\rho + \frac{1}{2}\hat{a}_z$$

2

Notice that $\|N\| = 1$ in any coordinates.

Angle that A makes with the normal is α such that

$$\cos\alpha = \frac{A \cdot N}{\|A\|\,\|N\|} = \frac{\langle 6, \sqrt{3}, -4\sqrt{3}\rangle \cdot \langle \frac{\sqrt{3}}{2}, 0, \frac{1}{2}\rangle}{\sqrt{87}} \quad 2$$

$$= \frac{\sqrt{3}}{\sqrt{87}} = \frac{1}{\sqrt{29}} \Rightarrow \alpha = \cos^{-1}\frac{1}{\sqrt{29}}$$

Angle that A makes with the tangent plane is

$$\boxed{\beta = 90° - \alpha} \quad 1$$

Q2.

(a) Find the directional derivative of $T = r^2 \sin\theta \cos\varphi$ in the direction $3\hat{a}_x - 4\hat{a}_z$ at the point $P(1, \frac{\pi}{6}, \frac{\pi}{2})$. **[10 pts]**

(b) Find $\nabla^2 V$ where $V = \rho z \cos 2\varphi$ in cylindrical coordinates where $\rho \neq 0$. **[7 pts]**

(c) Express ∇V in ~~spherical~~ coordinates. (Cartesian) **[8 pts]**

Sol. (a) The point P is clearly given in spherical coordinates in the spherical coordinates

$$\nabla T = \frac{\partial T}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial T}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \varphi}\vec{a}_\varphi \qquad 2$$

$$= 2r\sin\theta\cos\varphi\,\vec{a}_r + r\cos\theta\cos\varphi\,\vec{a}_\theta - r\sin\varphi\,\vec{a}_\varphi$$

$$\nabla T(1, \tfrac{\pi}{6}, \tfrac{\pi}{2}) = -\vec{a}_\varphi = \langle 0, 0, -1\rangle \qquad 1$$

Direction in spherical coordinates:

$$\begin{pmatrix} A_r \\ A_\theta \\ A_\varphi \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & \sqrt{3}/2 \\ 0 & \sqrt{3}/2 & -1/2 \\ -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix} = \begin{pmatrix} -2\sqrt{3} \\ 2 \\ -3 \end{pmatrix} \qquad 4$$

Directional derivative is

$$\frac{dT}{dA}(1, \tfrac{\pi}{6}, \tfrac{\pi}{3}) = \frac{\nabla T \cdot A}{\|A\|} = \frac{\langle 0,0,-1\rangle \cdot \langle -2\sqrt{3}, 2, -3\rangle}{5} \qquad 3$$

$$= \frac{3}{5}.$$

(b) $$\nabla V = \frac{\partial V}{\partial \rho}\vec{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \varphi}\vec{a}_\varphi + \frac{\partial V}{\partial z}a_z \qquad 3.5$$

$$= z\cos 2\varphi\,\vec{a}_\rho - 2z\sin 2\varphi\,\vec{a}_\varphi + \rho\cos 2\varphi\,\vec{a}_z$$

$$= \langle z\cos 2\varphi, -2z\sin 2\varphi, \rho\cos 2\varphi\rangle.$$

$$\nabla^2 V = \nabla\cdot\nabla V = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho z\cos 2\varphi) + \frac{1}{\rho}\frac{\partial}{\partial \varphi}(-2z\sin 2\varphi) + \frac{\partial}{\partial z}(\rho\cos 2\varphi) \qquad 3.5$$

$$= \frac{z\cos 2\varphi}{\rho} - \frac{4z\cos 2\varphi}{\rho} = \frac{-3z\cos 2\varphi}{\rho}$$

(c) ∇V in cartesian coordinates

$$\nabla V = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z\cos 2\varphi \\ -2z\sin 2\varphi \\ \rho\cos 2\varphi \end{pmatrix}$$

$$= \begin{pmatrix} z\cos\varphi\cos 2\varphi + 2z\sin\varphi\sin 2\varphi \\ z\sin\varphi\cos 2\varphi - 2z\cos\varphi\sin 2\varphi \\ \rho\cos 2\varphi \end{pmatrix} \quad 2$$

$$= \begin{pmatrix} z\cos\varphi(\cos^2\varphi - \sin^2\varphi) + 4z\sin^2\varphi\cos\varphi \\ z\sin\varphi(\cos^2\varphi - \sin^2\varphi) - 4z\sin\varphi\cos^2\varphi \\ \rho(\cos^2\varphi - \sin^2\varphi) \end{pmatrix} \quad 2$$

We know that $\cos\varphi = \frac{x}{\rho}$, $\sin\varphi = \frac{y}{\rho}$. So,

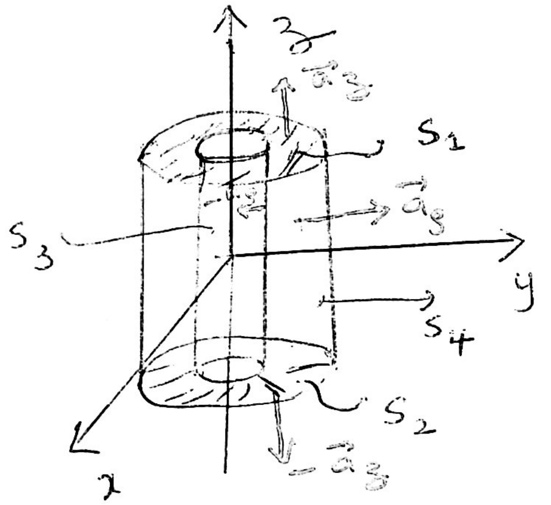
$$\nabla V = \begin{pmatrix} \frac{zx}{\rho}\left(\frac{x^2-y^2}{\rho^2}\right) + \frac{4z\,xy^2}{\rho^3} \\ \frac{zy}{\rho}\left(\frac{x^2-y^2}{\rho^2}\right) - \frac{4zyx^2}{\rho^3} \\ \frac{x^2-y^2}{\rho} \end{pmatrix} \quad 2$$

$$= \begin{pmatrix} \frac{zx^3 + 3xy^2z}{\rho^3} \\ -\frac{3x^2yz + y^3z}{\rho^3} \\ \frac{x^2-y^2}{\rho} \end{pmatrix} \quad 1 \quad = \begin{pmatrix} \frac{zx^3 + 3xy^2z}{(x^2+y^2)\sqrt{x^2+y^2}} \\ -\frac{zy^3 + 3x^2yz}{(x^2+y^2)\sqrt{x^2+y^2}} \\ \frac{x^2-y^2}{\sqrt{x^2-y^2}} \end{pmatrix} \quad 1$$

$\boxed{\text{(-3) for } \sin(\tan^{-1}\frac{y}{x}) \text{ etc.}}$

Q3. Verify the divergence theorem for the function $\boldsymbol{E} = 2\rho z^2 \hat{\boldsymbol{a}}_\rho + \rho \cos^2\varphi \, \hat{\boldsymbol{a}}_z$, over region defined by $2 < \rho < 5, -1 < z < 1, 0 < \varphi < 2\pi$. *[30 points]*

The closed surface S consists of 4 parts:



(1) S_1: Top given by $z = 1$

(1) S_2: bottom given by $z = -1$

$2 \le \rho \le 5, \quad 0 \le \varphi \le 2\pi$

(1) S_3: interior cylinder

$\rho = 2, \quad -1 \le z \le 1, \quad 0 \le \varphi \le 2\pi$

(1) S_4: external cylinder $\rho = 5, \quad -1 \le z \le 1, \quad 0 \le \varphi \le 2\pi$.

Divergence theorem

$$\int_S E \cdot ds = \int_V \operatorname{div} E \, dv \quad (1)$$

on S_1: $\quad ds = \rho \, d\rho \, d\varphi \, \vec{a}_z$ (1) $\Rightarrow \displaystyle\int_{S_1} E \cdot ds = \int_0^{2\pi} \int_2^5 \rho^2 \cos^2\varphi \, d\rho \, d\varphi$ (1)

$$\int_{S_1} E \cdot ds = \int_0^{2\pi} \frac{\cos 2\varphi + 1}{2} \, d\varphi \left[\frac{\rho^3}{3} \right]_{\rho=2}^{\rho=5} = 39 \cdot \frac{2\pi}{2} = \underline{\underline{39\pi}}$$

(2) (1) (6)

on S_2: $\quad ds = \rho \, d\rho \, d\varphi (-\vec{a}_z)$ ... (3)

So $\displaystyle\int_{S_2} E \cdot ds = -39\pi$.

on S_3: $ds = \rho \, d\varphi \, dz (-\vec{a}_\rho) = -2 \, d\varphi \, dz$ 1

So, $$\int_{S_3} E \cdot ds = \int_{-1}^{1} \int_{0}^{2\pi} -8 z^2 \, d\varphi \, dz \quad 1$$

$$= -16\pi \left. \frac{z^3}{3} \right]_{z=-1}^{z=1} \quad 1.5 \quad = -\frac{32}{3}\pi . \quad 1.5$$

on S_4: $ds = 5 \, d\varphi \, dz \, \vec{a}_\rho$ 1

So, $$\int_{S_4} E \cdot ds = \int_{-1}^{1} \int_{0}^{2\pi} 50 z^2 d\varphi \, dz \quad 1 \quad = 100\pi \left. \frac{z^3}{3} \right]_{-1}^{1} = \frac{200\pi}{3} \quad 1$$

Thus $$\oint_S E \cdot ds = 39\pi - 39\pi - \frac{32\pi}{3} + \frac{200\pi}{3} = \boxed{56\pi} \quad 2$$

$\nabla \cdot E = 4 z^2$ 1

So, $$\int_{-1}^{1} \int_{0}^{2\pi} \int_{2}^{5} 4 \rho z^2 \, d\rho \, d\varphi \, dz \quad 3$$

$$= 8\pi \left. \frac{\rho^2}{2} \right|_{\rho=2}^{\rho=5} \cdot \left. \frac{z^3}{3} \right|_{z=-1}^{z=1} \quad 2$$

$$= \frac{4\pi}{3} (25-4) \cdot (1+1) = \boxed{56\pi} \quad 1$$

Q~~5~~4 Let $\boldsymbol{E} = (20\rho\sin\varphi + 6z)\hat{\boldsymbol{a}}_\rho + 10\,\rho\cos\phi\,\hat{\boldsymbol{a}}_\varphi + 6\rho\,\hat{\boldsymbol{a}}_z$ be the electric field on a certain region of space.

(a) Verify that $\boldsymbol{E}$ is a conservative field. *[5 points]*

(b) Find the electric potential function V. *[1 ~~5~~ points]*

(c) Given two points $A(1,0,1)$ and $B(4,\pi/6,0)$ inside this region, find the electric potential at A, i.e. $V(A)$, given that $V(B) = 5$. *[5 points]*

Sol.

(a) $$\nabla \times E = \frac{1}{\rho}\begin{vmatrix} \vec{a}_\rho & \rho\,\vec{a}_\varphi & \vec{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\varphi} & \frac{\partial}{\partial z} \\ 20\rho\sin\varphi + 6z & 10\rho^2\cos\varphi & 6\rho \end{vmatrix}$$ 2

2 $$= \frac{1}{\rho}\left[(0-0)\vec{a}_\rho - (6-6)\rho\,\vec{a}_\varphi + (20\rho\cos\varphi - 20\rho\cos\varphi)\vec{a}_z\right]$$

$= \vec{0}$ 0.5. So, E is conservative 0.5

(b) A potential is given by
$-\nabla V = E \iff \left\langle \frac{\partial V}{\partial\rho}, \frac{1}{\rho}\frac{\partial V}{\partial\varphi}, \frac{\partial V}{\partial z}\right\rangle = -E$ 1

$$\frac{\partial V}{\partial\rho} = -20\rho\sin\varphi - 6z \quad 1$$

$$\Rightarrow V = -10\rho^2\sin\varphi - 6\rho z + g(\varphi, z) \quad 1.5$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial V}{\partial\varphi} = -10\rho\cos\varphi - 6z + \frac{1}{\rho}\frac{\partial g}{\partial\varphi} = -10\rho\cos\varphi - 6z$$ 1

$$\Rightarrow \frac{1}{\rho}\frac{\partial g}{\partial\varphi} = 0 \; 0.5 \Rightarrow g(\varphi, z) = f(z) \quad 0.5$$

$$\Rightarrow V = -10\rho^2\sin\varphi - 6\rho z + f(z) \quad 1$$

$$\Rightarrow \frac{\partial V}{\partial z} = -6\rho + f'(z) = -6\rho \; 1 \Rightarrow f'(z) = 0 \Rightarrow f(z) = c. \quad 1$$

$$\Rightarrow \boxed{V = -10\rho^2 \sin\varphi - 6\rho z + C} \quad 1.5$$

(c) $V(B) = V(4, \frac{\pi}{6}, 0) = -80 + C = 5$ 2

$\Rightarrow \quad C = 85$ 1

$V(A) = V(1, 0, 1) = 0 - 6 + 85 = 79.$ 2