Q1. (a) Determine whether, $S = \{\langle x, y, z, t\rangle \mid xy = \cancel{z}t\}$ is a subspace of $\mathbb{R}^4$? **[5 points]**

(b) Show that $S = \{\langle x, y, z, w\rangle \mid x + 2y = z + 3w = 0\}$ is a subspace of $\mathbb{R}^4$. **[5 points]**

(c) Find a basis and for the subspace, $S = \{\langle x, y, z, w\rangle \mid x + 2y = z + 3w = 0\}$ of $\mathbb{R}^4$. What is the dimension of this subspace? **[6 points]**

Counter example

(a) $\langle 1,1,1,1\rangle \in S$, $\langle 2,2,4,1\rangle \in S$ 2

$\langle 1,1,1,1\rangle + \langle 2,2,4,1\rangle = \langle 3,3,5,2\rangle \notin S$

$3\cdot 3 \neq 5\cdot 2$ 2

Second condition is not satisfied. S is not a subspace. 1

(b) $\langle -2y, y, -3w, w\rangle$ is the general form of a vector in V. 1+1

① $w = y = 0 \Rightarrow \vec{0} \in V$ 1

② $\langle -2y_1, y_1, -3w_1, w_1\rangle + \langle -2y_2, y_2, -3w_2, w_2\rangle$ $\in V$ $\in V$

$= \langle -2(y_1+y_2), y_1+y_2, -3(w_1+w_2), w_1+w_2\rangle$ 1

The components of the sum satisfy the equations.

Hence, V is closed under sum.

③ $k\langle -2y, y, -3w, w\rangle = \langle -2ky, ky, -3kw, kw\rangle$

The components of scalar multiplication satisfy the equations

$-2ky = 2(ky)$ $-3kw + 3kw = 0$ 1

V is closed under scalar multiplication. Hence, V is a subspace. 1

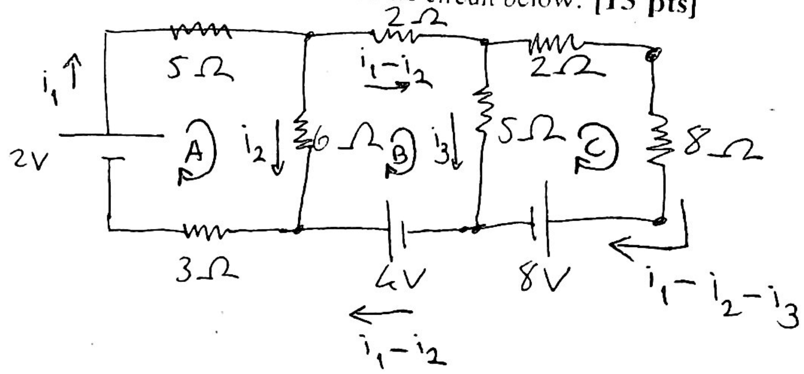
(c) $\langle -2y, y, -3w, w\rangle = y\langle -2, 1, 0, 0\rangle + w\langle 0, 0, -3, 1\rangle$ 2

$B = \{\langle -2,1,0,0\rangle, \langle 0,0,-3,1\rangle\}$ spans V. They are linearly independent 1+1 as they are not multiple of each other. B is a basis

$\dim V = 2$

Q2. Using Gaussian elimination, find the currents in all branches of the circuit below. **[15 pts]**

3



A: $-5i_1 - 6i_2 - 3i_1 + 2 = 0$ $\qquad 8i_1 + 6i_2 = 2$

2+1 B: $-2(i_1 - i_2) - 5i_3 + 6i_2 = -4$ $\qquad 2i_1 - 8i_2 + 5i_3 = 4$ $\qquad$ 1

C: $-10(i_1 - i_2 - i_3) - 8 + 5i_3 = 0$ $\qquad -10i_1 + 10i_2 + 15i_3 = 8$

$$\left(\begin{array}{ccc|c} 8 & 6 & 0 & 2 \\ 2 & -8 & 5 & 4 \\ -10 & 10 & 15 & 8 \end{array}\right) \xrightarrow{5R_2 + R_3} \left(\begin{array}{ccc|c} 8 & 6 & 0 & 2 \\ 2 & -8 & 5 & 4 \\ 0 & -30 & 40 & 28 \end{array}\right)$$

2 $\qquad$ 1

$\boxed{-4R_2}$ then $\boxed{R_1 + R_2}$

$$\left(\begin{array}{ccc|c} 8 & 6 & 0 & 2 \\ 0 & 38 & -20 & -14 \\ 0 & -30 & 40 & 28 \end{array}\right) \xrightarrow{\frac{1}{2}R_1,\ \frac{1}{2}R_2,\ \frac{1}{2}R_3} \left(\begin{array}{ccc|c} 4 & 3 & 0 & 1 \\ 0 & 19 & -10 & -7 \\ 0 & -15 & 20 & 14 \end{array}\right) \xrightarrow{\boxed{\frac{19}{15}R_3} \text{ then } \boxed{R_2 + R_3}} \left(\begin{array}{ccc|c} 4 & 3 & 0 & 1 \\ 0 & 19 & -10 & -7 \\ 0 & 0 & \frac{46}{3} & \frac{161}{15} \end{array}\right)$$

1 $\qquad$ 1 $\qquad$ 1

$4x_1 + 3x_2 = 1$

$19x_2 - 10x_3 = -7$

$\frac{46}{3}x_3 = \frac{161}{15}$

$$\boxed{\begin{array}{l} x_1 = 1/4 \\ x_2 = 0 \\ x_3 = 7/10 \end{array}}$$

3 $\qquad -\frac{10}{4} + \frac{21}{2}$

Q3. (a) Consider the system of non-homogenous linear algebraic equations,

$$\begin{aligned} ax_1 \qquad\qquad &+ x_3 = 161 \\ -x_1 + ax_2 \qquad &= 302 \\ x_2 &+ x_3 = 2016 \end{aligned}$$

If there is no parameters in the solution of the consistent system, ~~find a.~~ what are the values that a cannot take? *[10 points]*

(b) Consider the matrix

$$B = \begin{pmatrix} 3 & 6 & -1 & -5 & 5 \\ 2 & 4 & -1 & -3 & 2 \\ 3 & 6 & -2 & -4 & 1 \end{pmatrix}$$

If the system $BX = C$ is consistent. How many parameters does the solution have?
Find rank$(B|C)$.

[10 points]

(a) If there is no parameters then rank of $\begin{pmatrix} a & 0 & 1 \\ -1 & a & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is 3. 2

$$\begin{vmatrix} a & 0 & 1 \\ -1 & a & 0 \\ 0 & 1 & 1 \end{vmatrix} \neq 0 \qquad a^2 + (-1) \neq 0$$

$$a^2 - 1 \neq 0 \qquad a \neq \mp 1 \qquad 8$$

Or use echelon form.

(b)
$$\begin{pmatrix} 3 & 6 & -1 & 5 & 5 \\ 2 & 4 & -1 & 3 & 2 \\ 3 & 6 & -2 & 4 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - \frac{2}{3}R_1} \begin{pmatrix} 3 & 6 & -1 & 5 & 5 \\ 0 & 0 & -1/3 & -1/3 & -4/3 \\ 0 & 0 & -1 & -1 & -4 \end{pmatrix}$$

2 2

$$\xrightarrow{R_3 - 3R_2} \begin{pmatrix} 3 & 6 & -1 & 5 & 5 \\ 0 & 0 & -1/3 & -1/3 & -4/3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2

There are 3 free variables. Hence 3 parameters.

$\boxed{\text{Rank}(B|C) = 2}$ 2

If Rank$(B|C) = 3$, the system will be inconsistent. 2

Rank$(B|C)$ cannot be less than rank B.

Q4. (a) Find the eigenvalues of the matrix,

$$A = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{pmatrix}$$

[10 points]

(b) Find a nonzero vector v such that $A^{302}v = v$.

[10 points]

(a) $\begin{vmatrix} 3-\lambda & 0 & 0 \\ -4 & 6-\lambda & 2 \\ 16 & -15 & -5-\lambda \end{vmatrix} = -\lambda(\lambda-1)(\lambda-3) = 0$] 4 for all steps

3

$\Rightarrow \lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = 3.$ 3

(b) If v is the eigenvector of A, then $Av = v$.

Hence, $A^2 v = Av = v \Rightarrow A^2 v = v \Rightarrow A^3 v = Av = v$

$\Rightarrow A^3 v = v.$

$\vdots$

$A^{302} v = v$

2

We can choose v as the eigenvector of 1.

$\lambda_1 = 1 \quad \begin{pmatrix} 2 & 0 & 0 \\ -4 & 5 & 2 \\ 16 & -15 & -6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2/5 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 0 \\ -2/5 \\ 1 \end{pmatrix}.$

2 2 2

2

$v_1 = 0$

$v_2 = -\frac{2}{5} v_3$ 2

Q5. (a) Consider

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find matrices P and D, with P orthogonal and D diagonal, such that $A = PDP^{-1}$. [~~10~~ 11 points]

(b) Find the eigenvalues and eigen vectors of A^{-1}. [~~5~~ 6 points]

(a)

$$\begin{vmatrix} -\lambda & -1 & 1 \\ -1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+1)(\lambda-2) = 0$$

1 (showing all work)

~~0.5~~ 1

$\Rightarrow \lambda = \pm 1, 2$ are eigenvalues ~~0.5~~ 1

$\lambda_1 = 1$ $\begin{pmatrix} -1 & -1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $v_1 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ 2

$\lambda_2 = -1$ $\begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} -2/\sqrt{6} \\ -1/\sqrt{6} \\ 1/\sqrt{2} \end{pmatrix}$ 2

$\lambda_3 = 2$ $\begin{pmatrix} -2 & -1 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $v = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ $v_3 = \begin{pmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ 2

$$P = \begin{pmatrix} 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \quad D = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 2 \end{pmatrix}$$

1 1

(b) $\lambda_1 = 1$ $e_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ ~~1.5~~ 2 $\lambda_2 = -1$ $e_2 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$ ~~1.5~~ 2

$\lambda_3 = 1/2$ $e_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ 2

Q6. Find the inverse of the matrix A, if it exists, using the augmented matrix method where

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 3 & 0 & 1 \\ 1 & -1 & 2 \end{pmatrix}$$

Show the step by step calculations for full marks.

[10 pts]

$$\left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 0 & 6 & -5 & -3 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$

1 2

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & -1 & 0 & 2 \\ 0 & 0 & -5 & 3 & 1 & -6 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 2/5 & -2/5 \\ 0 & 0 & -5 & 3 & 1 & -6 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right)$$

2 1

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & 2/5 & -2/5 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -3/5 & -1/5 & 6/5 \end{array}\right)$$ 2

$$\to \quad A^{-1} = \begin{pmatrix} \frac{1}{5} & 2/5 & -2/5 \\ -1 & 0 & 1 \\ -3/5 & -1/5 & 6/5 \end{pmatrix}$$ 2