King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Final Exam
The First Semester of 2016-2017 (161)

Time Allowed: 180 Minutes

Name: SOLUTION ID#: ______

Instructor: ______ Sec #: ______ Serial #: ______

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear to get full credit.

Question #	Marks	Maximum Marks
1		24
2		22
3		22
4		25
5		25
6		22
Total		140

Q:1(18+6 points) Consider the Sturm–Liouville problem

$$x^2y'' + 3xy' + \lambda\, y = 0 \text{ with } y(1) = 0,\ y(e) = 0.$$

(a) Find eigenvalues and eigenfunctions of the problem.

(b) Put the differential equation in self–adjoint form and write its weight function.

(a) $m(m-1) + 3m + \lambda = 0$

$m^2 + 2m + \lambda = 0$

$m = -1 \pm \sqrt{1-\lambda}$ 3

Case I $\lambda = 1$, $m = -1, -1$

$y(x) = (c_1 + c_2 \ln x)x^{-1}$

$y(1) = 0 \Rightarrow c_1 = 0$

$y(e) = 0 \Rightarrow c_2 = 0$

Trivial solution. 3

Case II $\lambda < 1$

$m = -1 \pm \alpha$, $\alpha = \sqrt{1-\lambda}$

$y(x) = c_1 x^{-1+\alpha} + c_2 x^{-1-\alpha}$

$y(1) = 0 \Rightarrow c_1 + c_2 = 0$

$y(e) = 0 \Rightarrow c_1(e^{-1+\alpha} - e^{-1-\alpha}) = 0$

$2c_1 e^{-1}\dfrac{(e^{\alpha} - e^{-\alpha})}{2} = 0$

$2c_1 e^{-1} \sinh\alpha = 0$

$\Rightarrow c_1 = 0$ & $c_2 = 0$

Since $\alpha \neq 0$ 4

Case III $\lambda > 1$, let $\sqrt{\lambda - 1} = \alpha$

$m = -1 \pm \alpha i$

$y(x) = x^{-1}[c_1 \cos(\alpha \ln x) + c_2 \sin(\alpha \ln x)]$

$y(1) = 0 \Rightarrow c_1 = 0$ 4

$y(e) = 0 \Rightarrow c_2 e^{-1} \sin(\alpha) = 0$

Let $c_2 \neq 0$, then $\sin\alpha = 0$

$\Rightarrow \alpha = n\pi$

$\sqrt{\lambda - 1} = n\pi$

4 $\lambda = n^2\pi^2 + 1$

$y(x) = x^{-1}\sin(n\pi \ln x)$

(b) $y'' + \frac{3}{x}y' + \frac{\lambda}{x^2}y = 0$ 2

$IF = e^{\int \frac{3}{x}dx} = e^{3\ln x}$

$= x^3$ 2

$x^3y'' + 3x^2y' + x\lambda y = 0$

$\frac{d}{dx}[x^3y'] + \lambda x y = 0$

$w(x) = x$ 2

Q:2 (22 points) Find temperature $u(x,t)$ in a rod of length π by solving:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad 0 < x < \pi, \; t > 0,$$

$$u(0,t) = 0, \quad u(\pi,t) = 0, \quad t > 0$$

$$u(x,0) = f(x) = \begin{cases} 0 & 0 < x < \frac{\pi}{2} \\ x & \frac{\pi}{2} < x < \pi. \end{cases}$$

Let $u(x,t) = X(x)T(t)$

Then $X''T = XT'$

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda \quad ②$$

$$X'' + \lambda X = 0$$

$u(0,t) = 0 \Rightarrow X(0) = 0$

$u(\pi,t) = 0 \Rightarrow X(\pi) = 0$ ②

Case I $\quad m^2 = 0$ for $\lambda = 0$

$X(x) = C_1 + C_2 x$

$X(0) = 0 \Rightarrow C_1 = 0$

$X(\pi) = 0 \Rightarrow C_2 = 0$ ②

Case II $\quad \lambda = -\alpha^2 < 0$

$m = \pm\alpha$

$X(x) = C_1 \cosh\alpha x + C_2 \sinh\alpha x$

$X(0) = 0 \Rightarrow C_1 = 0$

$X(\pi) = 0 \Rightarrow C_2 \sinh\alpha\pi = 0$

$\Rightarrow C_2 = 0, \; \alpha \neq 0$ ③

Case III $\quad \lambda = \alpha^2 > 0$

$m = \pm\alpha i$

$X(x) = C_1 \cos\alpha x + C_2 \sin\alpha x$

$X(0) = 0 \Rightarrow C_1 = 0$

$X(\pi) = 0 \Rightarrow C_2 \sin\alpha\pi = 0$

Let $C_2 \neq 0$, $\sin\alpha\pi = 0$

$\Rightarrow \alpha\pi = n\pi$

$\alpha = n$

$\lambda_n = n^2$

$X_n(x) = C_2 \sin nx$ ⑤

$T' + n^2 T = 0$

$T_n(t) = C_3 e^{-n^2 t}$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin nx \, e^{-n^2 t} \quad ②$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

$$A_n = \frac{2}{\pi}\int_{\pi/2}^{\pi} x \sin nx \, dx \quad ②$$

$$= \frac{2}{\pi}\left(-x\frac{\cos nx}{n}\Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \frac{\cos nx}{n}\,dx \right)$$

$$= \frac{-2}{n\pi}\left((-1)^n \pi - \frac{\pi}{2}\cos\frac{n\pi}{2} \right) + \frac{2}{n^2\pi} \sin nx \Big|_{\pi/2}^{\pi}$$

$$= \frac{2(-1)^{n+1}}{n} + \frac{\cos n\pi/2}{n} + \frac{2}{n^2\pi}\sin\frac{n\pi}{2} \quad ②$$

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2(-1)^{n+1}}{n} + \frac{\cos\frac{n\pi}{2}}{n} - \frac{2\sin\frac{n\pi}{2}}{n^2\pi} \right) \sin nx \, e^{-n^2 t}$$

②

Q:3 (22 points) Use Laplace transform to solve the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad 0 < x < 1,\ t > 0,$$

subject to the boundary and initial conditions

$$\begin{aligned} u(0,t) &= 0, \quad u(1,t) = 0, \quad t > 0, \\ u(x,0) &= 0, \quad u_t(x,0) = 3\cos 2\pi x, \quad 0 < x < 1. \end{aligned}$$

$$\frac{d^2U(x,s)}{dx^2} = s^2U(x,s) - sU(x,0) - U_t(x,0) \quad (4)$$

$$\frac{d^2U}{dx^2} - s^2U = 3\sin 2\pi x \quad (2)$$

$$U_c = C_1\cosh sx + C_2\sinh sx \quad (2)$$

Let $U_p = A\sin 2\pi x$, then

$$\frac{d^2U_p}{dx^2} = -4\pi^2 A\sin 2\pi x$$

$$\Rightarrow -4\pi^2 A\sin 2\pi x - s^2 A\sin 2\pi x = 3\sin 2\pi x$$

$$\Rightarrow A = \frac{-3}{s^2+4\pi^2}$$

$$U_p = \frac{-3}{s^2+4\pi^2}\sin 2\pi x \quad (4)$$

$$U(x,s) = C_1\cosh sx + C_2\sinh sx - \frac{3}{s^2+4\pi^2}\sin 2\pi x \quad (2)$$

$$u(0,t) = 0 \Rightarrow U(0,s) = 0$$

$$\Rightarrow C_1 = 0 \quad (2)$$

$$u(1,t) = 0 \Rightarrow U(1,s) = 0$$

$$\Rightarrow C_2\sinh s = 0$$

$$C_2 = 0 \qquad s > 0$$

$$U(x,s) = -\frac{3}{s^2+4\pi^2}\sin 2\pi x \quad (3)$$

$$u(x,t) = \frac{-3}{2\pi}\sin 2\pi t\,\sin 2\pi x$$

(3)

Q:4 (25 points) Use separation of variables method to find steady state heat $u(x,z)$ is a right circular cylinder by solving the problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0,\ 0<r<1,\ 0<z<4,$$

subject to the boundary conditions

$$\begin{aligned} u(1,z) &= 0,\ 0<z<4 \\ u(r,0) &= 0,\ 0<r<1 \\ u(r,4) &= 4,\ 0<r<1 \end{aligned}$$

Also solution $u(r,z)$ is bounded at $r=0$.

Let $u(r,z) = R(r)\,Z(z)$

$$R''Z + \frac{1}{r}R'Z = -RZ''$$

$$\div RZ \quad \frac{R''}{R} + \frac{R'}{rR} = -\frac{Z''}{Z} = -\lambda \quad 2$$

$$R'' + \frac{1}{r}R' + \lambda R = 0$$

$$r^2R'' + rR' + (\lambda r^2 - 0)R = 0 \quad 2$$

Parametric Bessel eqn. with $\nu = 0$, $\lambda = \alpha^2$. 2

$$R(r) = C_1 J_0(\alpha r) + C_2 Y_0(\alpha r)$$

$Y_0(\alpha r) \to -\infty$ as $r \to 0^+$

$\Rightarrow C_2 = 0$ for Bounded sol. 2

$$R(r) = C_1 J_0(\alpha r)$$

$u(1,z) = 0 \Rightarrow R(1) = 0$

$\Rightarrow J_0(\alpha) = 0 \to$ ⊛

Case I with $h=0$, $b=1$

Let α_n are the solutions of ⊛ 2

So $R_n(r) = C_1 J_0(\alpha_n r)$ 2

For $\lambda = \alpha_n^2$, $Z'' - \alpha_n^2 Z = 0$

$$Z(z) = C_3 \cosh \alpha_n z + C_4 \sinh \alpha_n z$$

$u(r,0) = 0 \Rightarrow Z(0) = 0$

$\Rightarrow C_3 = 0$ 4

$$Z_n(z) = C_4 \sinh \alpha_n z$$

$$u(r,z) = \sum_{n=0}^{\infty} A_n \sinh \alpha_n z\, J_0(\alpha_n r)$$

$u(r,4) = 4 \Rightarrow$

$$4 = \sum_{n=0}^{\infty} A_n \sinh 4\alpha_n\, J_0(\alpha_n r)$$

$$A_n \sinh 4\alpha_n = \frac{2\cdot 4}{1 J_1^2(\alpha_n)} \int_0^1 r J_0(\alpha_n r)\,dr \quad 4$$

$$A_n = \frac{8}{J_1^2(\alpha_n)\sinh 4\alpha_n} \int_0^1 r J_0(\alpha_n r)\,dr$$

$$I = \frac{1}{\alpha_n^2}\int_0^{\alpha_n} t J_0(t)\,dt \qquad \alpha_n r = t$$

$$= \frac{1}{\alpha_n^2}\cdot \alpha_n J_1(\alpha_n) = \frac{J_1(\alpha_n)}{\alpha_n}$$

$$A_n = \frac{8}{\alpha_n J_1^2(\alpha_n)\sinh 4\alpha_n} \quad 5$$

Q:5 (25 points) Find the steady-state temperature $u(r,\theta)$ in a sphere of radius 1 by solving the problem

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta}=0,\quad 0<r<1,\ 0<\theta<\pi,$$

subject to the boundary condition

$$u(1,\theta) = 3\sin(\theta),\quad 0<\theta<\pi.$$

Find first ~~three~~ TWO nonzero terms in the final solution.

Let $u(r,\theta)=R(r)\Theta(\theta)$, then

$$R''\Theta+\frac{2}{r}R'\Theta=-\frac{1}{r^2}R\Theta''-\frac{\cot\theta}{r^2}R\Theta'$$

(2) $$\frac{r^2R''}{R}+2r\frac{R'}{R}=-\frac{\Theta''}{\Theta}-\frac{\cot\theta\,\Theta'}{\Theta}=\lambda$$

$$r^2R''+2rR'-\lambda R=0 \quad (2)$$

(2) $$\Theta''+\cot\theta\,\Theta'+\lambda\Theta=0 \rightarrow (*)$$

Let $x=\cos\theta$, then eqn. $(*)$ become

$$(1-x^2)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}+\lambda\Theta=0$$

This is a Legendre eqn. Solutions are for $\lambda=n(n+1)$, $n=0,1,2\ldots$

$$\Theta_n(\theta)=P_n(\cos\theta),\quad 0<\theta<\pi$$

(4)

For $\lambda=n(n+1)$,

$$r^2R''+2rR'-n(n+1)R=0$$

$$m^2-m+2m-n^2-n=0$$

$$m=n,\quad m=-(n+1)$$

$$R(r)=c_1r^n+c_2r^{-(n+1)}=c_1r^n+\frac{c_2}{r^{n+1}} \quad (4)$$

Sol. bounded as $r\to 0 \Rightarrow c_2=0$ (1)

$$u(r,\theta)=\sum_{n=0}^{\infty}A_nr^nP_n(\cos\theta) \quad (2)$$

(2) $u(1,\theta)=3\sin\theta$

$$\Rightarrow 3\sin\theta=\sum_{n=0}^{\infty}A_nP_n(\cos\theta) \quad (2)$$

$$A_n=\frac{2n+1}{2}\int_0^{\pi}3\sin\theta P_n(\cos\theta)\sin\theta\,d\theta$$

$$A_0=\frac{1}{2}\int_0^{\pi}3\sin^2\theta\,d\theta=\frac{3}{2}\int_0^{\pi}\frac{1-\cos 2\theta}{2}d\theta$$

$$=\frac{3}{4}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{\pi}=\frac{3\pi}{4} \quad (2)$$

$$A_1=\frac{3}{2}\cdot 3\int_0^{\pi}\sin^2\theta\cos\theta\,d\theta$$

$$=\frac{9}{2}\cdot\frac{\sin^3\theta}{3}=0 \quad (2)$$

$$A_2=\frac{15}{2}\int_0^{\pi}\sin^2\theta\cdot\frac{1}{2}(3\cos^2\theta-1)\,d\theta$$

(2) $\longrightarrow$

$$A_2 = \frac{15}{4}\int_0^{\pi} \sin^2\theta\,(3 - 3\sin^2\theta - 1)\,d\theta$$

$$= \frac{15}{4}\int_0^{\pi} 2\sin^2\theta\,d\theta - \frac{45}{4}\int_0^{\pi}(\sin^2\theta)^2\,d\theta$$

$$= \frac{15}{4}\int_0^{\pi} \frac{1-\cos 2\theta}{2}\,d\theta - \frac{45}{4}\int_0^{\pi}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{15}{8}\left(\theta - \frac{\sin 2\theta}{2}\right)\Big|_0^{\pi} - \frac{45}{8}\int_0^{\pi}(1 - 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$= \frac{15\pi}{8} - \frac{45}{8}\int_0^{\pi}\left[1 - 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right] d\theta$$

$$= \frac{15\pi}{8} - \frac{45}{8}\left[\theta - 2\frac{\sin 2\theta}{2} + \frac{1}{2}\theta + \frac{\sin 4\theta}{8}\right]_0^{\pi}$$

$$= \frac{15\pi}{8} - \frac{45}{8}\left[\frac{3\pi}{2}\right] = \frac{15\pi}{8} - \frac{135}{16}\pi$$

$$= \frac{30\pi - 135\pi}{16} = \frac{-105\pi}{16}$$

Q:6 (22 points) Use appropriate Fourier transform to solve

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,\quad 0<x<2,\ y>0,$$

subject to the conditions

$$u(0,y) = 0,\quad u(2,y)=e^{-y},\quad y>0$$

$$\left.\frac{\partial u}{\partial y}\right|_{y=0} = 0,\quad 0<x<2.$$

Apply $\mathcal{F}_c$ w.r.t. 'y' 3

$$\frac{d^2U(x,\alpha)}{dx^2}-\alpha^2U(x,\alpha)-u_y(x,0)=0 \quad 2$$

$$\frac{d^2U}{dx^2}-\alpha^2U=0 \quad 2$$

$$U(x,\alpha)=c_1\cosh\alpha x+c_2\sinh\alpha x \quad 2$$

$$u(0,y)=0 \Rightarrow U(0,\alpha)=0$$

$$\Rightarrow c_1=0 \quad 2$$

$$u(2,y)=e^{-y}$$

$$\Rightarrow U(2,\alpha)=\int_0^\infty e^{-y}\cos\alpha y\,dy$$

$$=\frac{1}{1+\alpha^2} \quad 3$$

$$\Rightarrow C_2\sinh 2\alpha=\frac{1}{1+\alpha^2} \quad 2$$

$$C_2=\frac{1}{(1+\alpha^2)\sinh 2\alpha}$$

$$U(x,\alpha)=\frac{\sinh\alpha x}{(1+\alpha^2)\sinh 2\alpha} \quad 4$$

$$u(x,y)=\frac{2}{\pi}\int_0^\infty\frac{\sinh\alpha x\cos\alpha y}{(1+\alpha^2)\sinh 2\alpha}\,d\alpha \quad 4$$