King Fahd University of Petroleum & Minerals
Department of Mathematics & Statistics
Math 301 Major Exam 1
The First Semester of 2016-2017 (161)

Time Allowed: 120 Minutes

Name: SOLUTION ID#: ____________

Instructor: ____________ Sec #: ________ Serial #: ________

- Mobiles and calculators are not allowed in this exam.
- Write all steps clear.

Question #	Marks	Maximum Marks
1		12
2		10
3		12
4		16
5		16
6		17
7		17
Total		100

Q:1 (6+6 points) Given the position vector $\vec{r}(t) = (\ln|\sec t| + t)\,\hat{\imath} + \cos t\,\hat{\jmath} - \sin t\,\hat{k}$ of a curve C. Find $\vec{r}''(0)$.

(b) Find the length of the curve $\vec{r}(t) = 2t^2\,\hat{\imath} + 3\cos(2t)\,\hat{\jmath} - 3\sin(2t)\,\hat{k}$ for $0 \le t \le 2\pi$.

(a) $\vec{r}\,'(t) = \left(\frac{\cancel{\sec t}\tan t}{\cancel{\sec t}} + 1\right)\hat{i} - \sin t\,\hat{j} - \cos t\,\hat{k}$ 2

$\vec{r}\,''(t) = \sec^2 t\,\hat{i} - \cos t\,\hat{j} + \sin t\,\hat{k}$ 2

$\vec{r}\,''(0) = \hat{i} - \hat{j} + 0\hat{k}$ 2

(b) $\vec{r}\,'(t) = 4t\,\hat{i} - 6\sin 2t\,\hat{j} - 6\cos 2t\,\hat{k}$ 2

$\|\vec{r}\,'(t)\| = \sqrt{16t^2 + 36}$ 2

$s = 2\int_0^{2\pi} \sqrt{4t^2 + 9}\,dt$ 2

Q:2 (10 points) Let $f(x,y) = xy - 3y^2$. Find the directional derivative of f at $(1,-1)$ in the direction of a tangent vector to the graph of $2x^2 + 3y^2 = 30$ at the point $(3,2)$.

$4x + 6yy' = 0 \qquad y' = -\frac{4x}{6y} = \frac{-2x}{3y}$

$m = y'\big|_{(3,2)} = \frac{-6}{6} = -\frac{1}{1}$ 2

A vector parallel to the tangent line is $\vec{u} = \pm(\hat{i} - \hat{j})$ and $\hat{u} = \pm\frac{(\hat{i}-\hat{j})}{\sqrt{2}}$ 2

$\nabla f = y\hat{i} + (x - 6y)\hat{j}$ 2

$\nabla f(1,-1) = -\hat{i} + 7\hat{j}$ 1

$D_{\hat{u}} f(1,-1) = \pm \frac{-1-7}{\sqrt{2}} = \mp \frac{8}{\sqrt{2}}$ 3

Q:3 (12 points) Let $\vec{F} = \sin(xy)\,\hat{\imath} - 2e^{xz}\,\hat{\jmath} + \ln(xy)\,\hat{k}$. Show that div(curl($\vec{F}$))=0.

(Show all your work)

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin(xy) & -2e^{xz} & \ln(xy) \end{vmatrix}$$

$$= \hat{i}\left(\frac{1}{xy}\cdot x + 2x e^{xz}\right) - j\left(\frac{1}{xy}\cdot y - 0\right) + \hat{k}\left(-2z e^{xz} - x\cos xy\right)$$

$$= \left(\frac{1}{y} + 2x e^{xz}\right)\hat{i} - \frac{1}{x}\hat{j} - \left(2z e^{xz} + x\cos xy\right)\hat{k}$$

6

$$\text{div}(\text{curl}\,\vec{F}) = \frac{\partial}{\partial x}\left(\frac{1}{y} + 2x e^{xz}\right) - \frac{\partial}{\partial y}\left(\frac{1}{x}\right) - \frac{\partial}{\partial z}\left(2z e^{xz} + x\cos xy\right)$$

$$= 0 + \cancel{2e^{xz}} + \cancel{2xz e^{xz}} - 0 - \cancel{2e^{xz}} - \cancel{2zx e^{xz}} + 0$$

6

$$= 0$$

Q:4 (16 points) Show that the integral is independent of path and then evaluate the integral

$$\int_{(1,0,2)}^{(2,\pi,3)} (2x\cos y + 2e^{3z})\,dx + (-x^2\sin y + 1)\,dy + (6xe^{3z} + 2)\,dz.$$

$P = 2x\cos y + 2e^{3z}$, $Q = 1 - x^2\sin y$, $R = 6xe^{3z} + 2$

$\frac{\partial P}{\partial y} = -2x\sin y = \frac{\partial Q}{\partial x}$, $\frac{\partial P}{\partial z} = 6e^{3z} = \frac{\partial R}{\partial x}$

$\frac{\partial Q}{\partial z} = 0 = \frac{\partial R}{\partial y}$. The integral is independent of path. 4

There exist $\phi(x,y,z)$ such that

$\frac{\partial \phi}{\partial x} = 2x\cos y + 2e^{3z}$, $\frac{\partial \phi}{\partial y} = 1 - x^2\sin y$, $\frac{\partial \phi}{\partial z} = 6xe^{3z} + 2$

$\Rightarrow \phi(x,y,z) = x^2\cos y + 2xe^{3z} + g(y,z)$

$\frac{\partial \phi}{\partial y} = -x^2\sin y + \frac{\partial g}{\partial y} = 1 - x^2\sin y$

$\Rightarrow \frac{\partial g}{\partial y} = 1 \Rightarrow g(y,z) = y + h(z)$

So $\phi(x,y,z) = x^2\cos y + 2xe^{3z} + y + h(z)$

$\frac{\partial \phi}{\partial z} = 6xe^{3z} + h'(z) = 6xe^{3z} + 2$

$\Rightarrow h'(z) = 2 \Rightarrow h(z) = 2z$

$\phi(x,y,z) = x^2\cos y + 2xe^{3z} + y + 2z$ 6

$$\int_{(1,0,2)}^{(2,\pi,3)} (2x\cos y + 2e^{3z})\,dx + (1 - x^2\sin y)\,dy + (6xe^{3z} + 2)\,dz$$

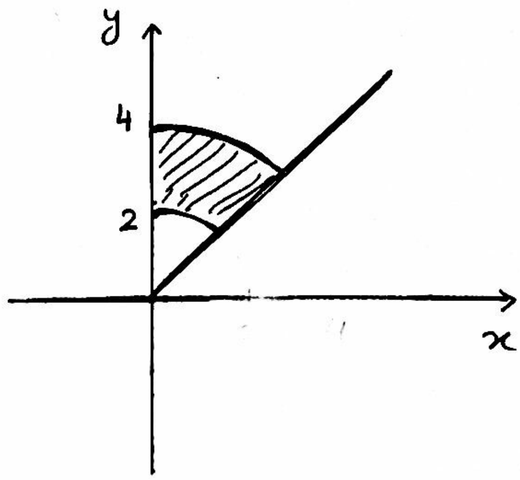
$= \phi(2,\pi,3) - \phi(1,0,2)$

$= (-4 + 4e^9 + \pi + 6) - (1 + 2e^6 + 0 + 4)$

$= 4e^9 - 2e^6 + \pi - 3$ 6

Q:5 (16 points) Use Green's theorem to evaluate the line integral

$\oint_C (2y^2 + \tan(2x^2))dx + (3x^2 + 4xy)dy$, where C is the positively oriented boundary of the region bounded by the graphs of $y = x$, $x = 0$, $x^2 + y^2 = 4$, $x^2 + y^2 = 16$ with $y \geq 0$.



$$= \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad 4$$

$$= \iint_R (6x + 4y - 4y)\, dA \qquad 2$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_2^4 6r\cos\theta \cdot r\, dr\, d\theta \qquad 4$$

$$= \sin\theta \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cdot \cancel{6}^{\,2} \frac{r^3}{\cancel{3}} \Big|_2^4 \qquad 3$$

$$= \left(1 - \frac{\sqrt{2}}{2}\right)(128 - 16) = \frac{(2 - \sqrt{2})}{2}(112)$$

$$= 56(2 - \sqrt{2}) \qquad 3$$

Q:6 (17 points) Use Stokes' theorem to evaluate the integral $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = 2xy\,\hat{\imath} - 3yz\,\hat{\jmath} + 4xz\hat{k}$ and C is the curve of intersection of the plane $2x - 3y + 4z = 10$ with the coordinate planes.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -3yz & 4xz \end{vmatrix} = \hat{i}\,(0+3y) - \hat{j}\,(4z-0) + \hat{k}\,(0-2x) = 3y\,\hat{i} - 4z\,\hat{j} - 2x\,\hat{k}$$

2

$$\hat{n} = \frac{2\hat{i} - 3\hat{j} + 4\hat{k}}{\sqrt{29}}$$ 2

$$ds = \sqrt{1 + f_x^2 + f_y^2}\, dA, \quad z = \frac{5}{2} - \frac{1}{2}x + \frac{3}{4}y$$

$$= \sqrt{1 + \frac{1}{4} + \frac{9}{16}}\, dA = \sqrt{\frac{16+4+9}{16}}\, dA = \frac{\sqrt{29}}{4}\, dA$$ 2

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot \hat{n}\, ds$$ 2

$$= \iint_R \frac{6y + 12z - 8x}{\sqrt{29}} \cdot \frac{\sqrt{29}}{4}\, dA$$ 2

$$= \int_0^{-10/3} \int_0^{5+\frac{3}{2}y} \left[6y + 12\left(\frac{5}{2} - \frac{1}{2}x + \frac{3}{4}y\right) - 8x\right] dx\, dy$$ 3

$$= \int_0^{-\frac{10}{3}} \int_0^{5+\frac{3}{2}y} (15y - 14x + 30)\, dx\, dy$$

$$= \int_0^{-\frac{10}{3}} \left[15y\left(5+\frac{3}{2}y\right) - 7\left(5+\frac{3}{2}y\right)^2 + 30\left(5+\frac{3}{2}y\right)\right] dy$$ 2

$$= \int_0^{-\frac{10}{3}} \left(75y + \frac{45}{2}y^2 - 175 - 105y - \frac{63}{4}y^2 + 150 + 45y\right) dy$$

$$= \left[15\frac{y^2}{2} + \frac{9}{4}y^3 - 25y\right]_0^{-\frac{10}{3}} = \frac{\cancel{15}^{5}}{\cancel{2}} \cdot \frac{\cancel{100}^{50}}{\cancel{9}_{3}} - \frac{\cancel{9}}{\cancel{4}} \cdot \frac{\cancel{1000}^{250}}{\cancel{27}_{3}} + \frac{250}{3}$$

$\boxed{\frac{250}{3}}$ 2

Q:7 (17 points) Use divergence theorem to evaluate $\iint\limits_S (\vec{F}.\hat{n})\, dS$

where $\vec{F} = x^3\,\hat{i} + y^3\,\hat{j} + z^3\,\hat{k}$ and D the region bounded by the surface S given by

$z = \sqrt{9 - x^2 - y^2}$, $z \geq 0$.

$$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3z^2 \qquad 2$$

$$\iint\limits_S \vec{F}\cdot\hat{n}\, ds = \iiint\limits_D 3(x^2+y^2+z^2)\, dv \qquad 3$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^3 3\rho^2 \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi \qquad 5$$

$$= 2\pi \cdot (-\cos\phi)\Big|_0^{\frac{\pi}{2}} \cdot 3\frac{\rho^5}{5}\Big|_0^3$$

243
$\times 3$
729

$$= 2\pi(0+1)\cdot\frac{729}{5} \qquad 5$$

$$= 2\pi \cdot \frac{729}{5}$$

$$= \frac{1458\pi}{5} \qquad 2$$

Q:7 $\quad div\,\vec{F} = 3x^2 + 3y^2 + 3z^2$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_D div\,\vec{F}\,dv$$

$$= \int_0^{2\pi}\int_0^{3}\int_0^{\sqrt{9-r^2}} (3r^2 + 3z^2)\, r\,dz\,dr\,d\theta$$

$$= 2\pi \int_0^3 \left[3r^3\sqrt{9-r^2} + r(9-r^2)^{\frac{3}{2}}\right] dr$$

$$= 6\pi \int_0^3 r^2\, r\,(9-r^2)^{\frac{1}{2}}\,dr + 2\pi\int_0^3 r\,(9-r^2)^{\frac{3}{2}}\,dr$$

$$= 6\pi\left[\frac{r^2}{-2}\,\frac{2}{3}(9-r^2)^{\frac{3}{2}}\Big|_0^3 - \int_0^3 \frac{2r}{-2}\,\frac{2}{3}(9-r^2)^{\frac{3}{2}}\,dr\right]$$

$$+ \frac{2\pi}{-2}\,\frac{2}{5}(9-r^2)^{\frac{5}{2}}\Big|_0^3$$

$$= 4\pi\left[0 + \frac{\frac{2}{5}(9-r^2)^{\frac{5}{2}}}{-2}\Big|_0^3\right. - \frac{2\pi}{5}\left(0 - 9^{\frac{5}{2}}\right)$$

$$= -\frac{4\pi}{5}\left(0 - 9^{\frac{5}{2}}\right) + \frac{2\pi}{5}\cdot 243$$

$$= \frac{4\pi}{5}\cdot 243 + \frac{2\pi}{5}\cdot 243$$

$$= \frac{972\pi + 486\pi}{5} = \frac{1458\pi}{5}$$