KFUPM T161

Dept. Math & Stat. Math 260 A.Y: 016/017

TEST N°: 4

Exercise 1

Solve the DE : $y'' - 4y' + 13y = 0$

Solution.

The characteristic equation of the DE is:

$$r^2 - 4r + 13 = 0 \iff (r-2)^2 - 4 + 13 = 0$$
$$\iff (r-2)^2 + 9 = 0 \iff (r-2-3i)(r-2+3i) = 0$$

So the roots are $r_1 = 2 + 3i$ and $r_2 = \overline{r_1}$.

Thus the G.S of the given DE is:

$$y = c_1 e^{2x}\cos(3x) + c_2 e^{2x}\sin(3x)$$

Exercise 2.

Find a 3^{rd} order DE with $y = Ae^{x} + Be^{2x} + Cxe^{2x} + e^{-x}$ as general solution.

Solution.

$y = Ae^{x} + Be^{2x} + Cxe^{2x} + e^{-x}$ is the G.S of a DE of the form $y''' + by'' + ay' + cy = f(x)$, with $y_c = Ae^{x} + Be^{2x} + Cxe^{2x}$ is the complementary sol. and $y_p = e^{-x}$ is a particular solution.

The roots of the characteristic equation of the associated homogeneous DE are

$$r_1 = 1, \quad r_2 = r_3 = 2.$$

Hence the char. equ is

$$(r-1)(r-2)(r-2) = 0 \iff (r-1)(r^2 - 4r + 4) = 0$$
$$\iff r^3 - 5r^2 + 8r - 4 = 0$$

Hence the Homogeneous DE is:

$$y^{(3)} - 4y^{(2)} + 8y' - 4y = 0$$

The required DE is

$$y^{(3)} - 4y^{(2)} + 8y' - 4y = f(x)$$

As $y_p = e^{-x}$ is a particular solution, we have

$$f(x) = y_p^{(3)} - 4y_p^{(2)} + 8y_p' - 4y_p$$
$$= -e^{-x} - 4e^{-x} + 8(-e^{-x}) - 4e^{-x}$$
$$= -17e^{-x}.$$

Therefore the required DE is:

$$y^{(3)} - 4y^{(2)} + 8y' - 4y = -17e^{-x}$$

<u>Exercise 3</u>. Use variation of parameters to find a particular solution of the DE

$$y'' + y = 2\cos x$$

<u>Solution.</u>

The characteristic equation of the associated homogeneous DE is:

$$r^2 + 1 = 0 \iff r = \pm i$$

The complementary solution is $y_c = c_1 \cos x + c_2 \sin x$

Let $y_1 = \cos x$, $y_2 = \sin x$; then

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

We seek a solution $y_p = u_1(x)\cos x + u_2(x)\sin x$

with $u_1'(x) = \dfrac{\begin{vmatrix} 0 & \sin x \\ 2\cos x & \cos x \end{vmatrix}}{W} = -2(\sin x)(\cos x)$

and $u_2'(x) = \dfrac{\begin{vmatrix} \cos x & 0 \\ -\sin x & 2\cos x \end{vmatrix}}{W} = 2\cos^2 x$

So $u_1(x) = \int -2(\sin x)(\cos x)\, dx$

$= -\int \sin(2x)\, dx = \frac{1}{2}\cos(2x) + c$, we take $u_1(x) = -\sin(2x)$

$u_2(x) = \int 2\cos^2 x\, dx = \int \frac{1+\cos(2x)}{2}\, dx$

$= \frac{x}{2} + \frac{1}{4}\sin(2x) + c$

Thus, just take $u_2(x) = \frac{x}{2} + \frac{1}{4}\sin(2x)$

It follows that $y_p(x) = \frac{1}{2}(\cos 2x)(\cos x) + \left(\frac{x}{2} + \frac{1}{4}\sin(2x)\right)\sin x$

Therefore, the G.S of the given DE is:

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2}(\cos x)(\cos 2x) + (\sin x)\left(\frac{x}{2} + \frac{1}{4}\sin(2x)\right)$$

Exercise 4.

Find the form of a particular solution of the following DE:

$$y^{(3)} - 2y^{(2)} + y' = e^x + x.$$

Solution.

The characteristic equation of the associated homogeneous DE is:

$$r^3 - 2r^2 + r = 0$$

$\Leftrightarrow r(r-1)^2 = 0$

So the complementary solution is

$$y_c = c_1 + c_2 e^x + c_3 x e^x$$

• The DE: $y^{(3)} - 2y^{(2)} + y' = e^x$ has a particular solution of the form $y_{p_1} = x^k A e^x$, where k is the smallest positive integer such that no term of y_{p_1} is duplicated in y_c. So $k = 2$.

that is to say $y_{p_1} = A x^2 e^x$.

• Now, the DE: $y^{(3)} - 2y^{(2)} + y' = x$ has a particular solution of the form $y_{p_2} = x^s(Cx + D)$, where s is the smallest positive integer such that no term of y_{p_2} is duplicated in y_c. Thus $s = 1$; and

$$y_{p_2} = x(Cx + D).$$

It follows that $y_p = y_{p_1} + y_{p_2} = A x^2 e^x + x(Cx + D)$ is the form of a particular solution of the given DE.

Exercise 5. Let $A = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{bmatrix}$

(1) Show that the characteristic polynomial of A is
$\det(\lambda I - A) = (\lambda - 1)^2(\lambda - 3)$.

(2) Diagonalize A.

(3) Use Cayley-Hamilton theorem to find A^{-1}, A^3, A^4.

Solution.

(1)
$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 3 & 2 & 0 \\ 0 & \lambda - 1 & 0 \\ 4 & -4 & \lambda - 1 \end{vmatrix} = (\lambda - 1)\begin{vmatrix} \lambda - 3 & 0 \\ 4 & \lambda - 1 \end{vmatrix}$$

$$= (\lambda - 1)(\lambda - 3)(\lambda - 1) = (\lambda - 1)^2(\lambda - 3)$$

(2) • The eig. vect associated with $\lambda_1 = 1$.

Let us solve the homogeneous Linear system $(A - I)K = 0$, where $K = [\alpha \ \beta \ \gamma]^T$.

$$\begin{cases} 2\alpha - 2\beta = 0 \\ -4\alpha + 4\beta = 0 \end{cases} \Longleftrightarrow \alpha = \beta. \text{ Thus } K = \begin{bmatrix} \alpha \\ \alpha \\ \gamma \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus the eig. space E_1 is spanned by the vectors

$U_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $U_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

• Eig. vect. assoc. with $\lambda = 3$

Let us solve the system $(A - 3I)K = 0$, where $K = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$:

$$\begin{cases} -2\beta = 0 \\ -2\beta = 0 \\ -4\alpha + 4\beta - 2\gamma = 0 \end{cases} \Longleftrightarrow \begin{cases} \beta = 0 \\ \gamma = -2\alpha \end{cases}$$

Thus $K = \begin{bmatrix} \alpha \\ 0 \\ -2\alpha \end{bmatrix} = \alpha . \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

Let $U_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$, and $P = [U_1 | U_2 | U_3] = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{bmatrix}$; then

$P^{-1} A P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

3 By Cayley-Hamilton theorem, we have

$$(A - 3I)(A^2 - 2A + I) = 0; \text{ that is}$$

$$A^3 - 5A^2 + 7A - 3I = 0.$$

Hence $I = A\left[\frac{1}{3} A^2 - \frac{5}{3} A + \frac{7}{3} I\right].$

It follows that A is invertible and

$$A^{-1} = \frac{1}{3} A^2 - \frac{5}{3} A + \frac{7}{3} I.$$

$$A^3 = 5A^2 - 7A + 3I.$$

$$A^4 = 5A^3 - 7A^2 + 3A$$
$$= 5(5A^2 - 7A + 3I) - 7A^2 + 3A$$
$$= 8A^2 + 32A + 15I$$

Exercise 6: Is the matrix $A=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ diagonalizable? why?

Solution.

- $p_A(\lambda) = \det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 \\ 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2$
- The eig. vect. asso. with $\lambda = 2$

Let us solve the system $(A - 2I)K = 0$, where $K = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$:

$$\begin{cases} 0\alpha + \beta = 0 \\ 0\alpha + 0\beta = 0 \end{cases} \iff \beta = 0.$$

So $K = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} = \alpha . \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

It follows that $E_2 = \mathrm{Span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, and $\dim(E_2) = 1$

As $\dim(E_2) = 1 \neq \mathrm{mul}(2) = 2$, the matrix A is not diagonalizable.