KFUPM

Dept. Math. & Stat

T161

Math 260

Quiz N° 3

Exercise 1:

(a) Use elementary row operations to find A^{-1}, where

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

(b) Use question (a) to solve the system:

$$\begin{cases} x + y + 2z = 2 \\ x + z = 3 \\ -x + y + z = 2 \end{cases}$$

Exercise 2.

Assume $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 1122.$

Evaluate the following determinant:

$$D = \begin{vmatrix} x_1 + 10 z_1 & 2 z_1 & 3 y_1 \\ x_2 + 10 z_2 & 2 z_2 & 3 y_2 \\ x_3 + 10 z_3 & 2 z_3 & 3 y_3 \end{vmatrix}$$

Exercise 3. Use Cramer's Rule to solve the system

$$\begin{cases} x + 4y + 3z = 1 \\ x + 4y + 5z = 0 \\ 2x + 5y + z = 1 \end{cases}$$

Exercise 4: Express the vector $\vec{t}$ as a linear combination of the vectors $\vec{u}, \vec{v}, \vec{w}$, where

$\vec{t} = (7, 7, 7)$, $\vec{u} = (2, 5, 3)$, $\vec{v} = (4, 1, -2)$, $\vec{w} = (1, 1, 5)$

Exercise 5: Show that $W = \{(x, y, z) \in \mathbb{R}^3 \mid z = -x + 7y\}$ is a subspace of $\mathbb{R}^3$.

Exercise 6: Is the set $W = \{(x, y) \in \mathbb{R}^2 \mid (x+y)^2 = x^2 + y^2\}$ a subspace of $\mathbb{R}^3$?

Sol. Ex 1:

(a)

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+2R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right] \xrightarrow[R_1-2R_3]{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & -4 & -2 \\ 0 & -1 & 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & -2 & 3 & 1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right] \xrightarrow{(-1)R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 2 & -3 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \end{array}\right]$$

It follows that $A^{-1} = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -3 & -1 \\ -1 & 2 & 1 \end{bmatrix}$.

(b) The matrix form of the system is $AX = B$, where

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$.

As A is nonsingular, we get $X = A^{-1}B = \begin{bmatrix} 1 & -1 & -1 \\ 2 & -3 & -1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 2-3-2 \\ 4-9-2 \\ -2+6+2 \end{bmatrix} = \begin{bmatrix} -3 \\ -7 \\ 6 \end{bmatrix}$$

Sol. Ex 2:

$$D = \begin{vmatrix} x_1+10z_1 & 2z_1 & 3y_1 \\ x_2+10z_2 & 2z_2 & 3y_2 \\ x_3+10z_3 & 2z_3 & 3y_3 \end{vmatrix} = (2\times 3)\begin{vmatrix} x_1+10z_1 & z_1 & y_1 \\ x_2+10z_2 & z_2 & y_2 \\ x_3+10z_3 & z_3 & y_3 \end{vmatrix}$$

$$\underline{\underline{C_1-10C_2}}\; 6\begin{vmatrix} x_1 & z_1 & y_1 \\ x_2 & z_2 & y_2 \\ x_3 & z_3 & y_3 \end{vmatrix} \;\underline{\underline{\text{swap}(C_2, C_3)}}\; (-6)\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= (-6)\times 1122 \; \boxed{= -6732.}$$

Sol. Ex 3:

$$D = \begin{vmatrix} 1 & 4 & 3 \\ 1 & 4 & 5 \\ 2 & 5 & 1 \end{vmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - R_1} \begin{vmatrix} 1 & 4 & 3 \\ 0 & 0 & 2 \\ 0 & -3 & -5 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ -3 & -5 \end{vmatrix} = 6$$

$$D_x = \begin{vmatrix} 1 & 4 & 3 \\ 0 & 4 & 5 \\ 1 & 5 & 1 \end{vmatrix} \xrightarrow{R_3 - R_1} \begin{vmatrix} 1 & 4 & 3 \\ 0 & 4 & 5 \\ 0 & 1 & -2 \end{vmatrix} = \begin{vmatrix} 4 & 5 \\ 1 & -2 \end{vmatrix} = -8-5 = -13$$

$$D_y = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 5 \\ 2 & 1 & 1 \end{vmatrix} \xrightarrow{R_3 - R_1} \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 5 \\ 1 & 0 & -2 \end{vmatrix} \xrightarrow{\text{Expand along } C_2} - \begin{vmatrix} 1 & 5 \\ 1 & -2 \end{vmatrix}$$
$$= -(-2-5)$$
$$= 7$$

$$D_z = \begin{vmatrix} 1 & 4 & 1 \\ 1 & 4 & 0 \\ 2 & 5 & 1 \end{vmatrix} \xrightarrow{R_1 - R_3} \begin{vmatrix} -1 & -1 & 0 \\ 1 & 4 & 0 \\ 2 & 5 & 1 \end{vmatrix} \xrightarrow[\text{along } C_3]{\text{Expand}} \begin{vmatrix} -1 & -1 \\ 1 & 4 \end{vmatrix}$$
$$= -4 + 1 = -3$$

Thus $x = \dfrac{D_x}{D} = -\dfrac{13}{6}$; $y = \dfrac{D_y}{D} = \dfrac{7}{6}$, $z = \dfrac{D_z}{D}$
$$= -\frac{1}{2}$$

Sol. Ex 4:

Let us find $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\vec{t} = \alpha \vec{u} + \beta \vec{v} + \gamma \vec{w}$; that is to say:

$$\begin{cases} 2\alpha + 4\beta + \gamma = 7 \\ 5\alpha + \beta + \gamma = 7 \\ 3\alpha - \beta + 5\gamma = 7 \end{cases}$$

It suffices to reduce the augmented matrix

$$M = \left[\begin{array}{ccc|c} 2 & 4 & 1 & 7 \\ 5 & 1 & 1 & 7 \\ 3 & -1 & 5 & 7 \end{array}\right]$$

Indeed, $M \xrightarrow[R_2 - R_3]{R_1 - R_3} \left[\begin{array}{ccc|c} -1 & 5 & -4 & 0 \\ 2 & 2 & -4 & 0 \\ 3 & -1 & 5 & 7 \end{array}\right] \xrightarrow[(1/2)R_2]{R_3 + R_1} \left[\begin{array}{ccc|c} -1 & 5 & -4 & 0 \\ 1 & 1 & -2 & 0 \\ 2 & 4 & 1 & 7 \end{array}\right]$

$$\xrightarrow[R_3-2R_2]{R_1+R_2}\left[\begin{array}{ccc|c}0&6&-6&0\\1&1&-2&0\\0&2&5&7\end{array}\right]\xrightarrow{(1/6)R_1}\left[\begin{array}{ccc|c}0&1&-1&0\\1&1&-2&0\\0&2&5&7\end{array}\right]$$

$$\xrightarrow[R_2-R_1]{R_3-2R_1}\left[\begin{array}{ccc|c}0&1&-1&0\\1&0&-1&0\\0&0&7&7\end{array}\right]\xrightarrow{(1/7)R_3}\left[\begin{array}{ccc|c}0&1&-1&0\\1&0&-1&0\\0&0&1&1\end{array}\right]$$

$$\xrightarrow[R_2+R_3]{R_1+R_3}\left[\begin{array}{ccc|c}0&1&0&1\\1&0&0&1\\0&0&1&1\end{array}\right]\xrightarrow{\text{swap}(R_1,R_2)}\left[\begin{array}{ccc|c}1&0&0&1\\0&1&0&1\\0&0&1&1\end{array}\right]$$

It follows that $\alpha=\beta=\gamma=1$; and consequently,

$$\boxed{\vec{t}=\vec{u}+\vec{v}+\vec{w}}$$

Sol. Ex 5.

- As $0=-0+7\times 0$, $0_{\mathbb{R}^3}=(0,0,0)\in W$.
- Let $\vec{u}=(x,y,z)$ and $\vec{v}=(a,b,c)$ be vectors of W; then

$z=-x+7y$ and $c=-a+7b$.

By adding the two equalities (side by side), we get

$z+c=-(x+a)+7(y+b)$; so $\vec{u}+\vec{v}=(x+a,y+b,z+c)\in W$

- Let $\vec{u}=(x,y,z)\in W$ and $\alpha\in\mathbb{R}$. Then $z=-x+7y$;

hence $(\alpha z)=-(\alpha x)+7(\alpha y)$. Consequently,

$$\alpha\vec{u}=(\alpha x,\alpha y,\alpha z)\in W$$

Therefore W is a subspace of $\mathbb{R}^3$.

Sol. Ex.6. Here, it is clear that $0=(0,0,0)\in W$; and if $\vec{u}\in W$ and $\alpha\in\mathbb{R}$, then $\alpha\vec{u}\in W$. But W is not closed under vector addition; Indeed $\vec{u}=(1,0)$, $\vec{v}=(0,1)$ are in W;

but $\vec{u} + \vec{v} = (1,1) \notin W$, as

$$(1+1)^2 = 4 \neq 1^2 + 1^2.$$

Therefore W is not a subspace of $\mathbb{R}^3$.