KFUPM

Dept. Math & Stat.

T161

Math 260

Quiz 2

①

EX1: Solve the DE: $\frac{dy}{dx} + \frac{2}{x} y = -(x^2 \cos x) y^2$.

solution.

This is a Bernoulli DE. We use the substitution

$v = y^{1-2} = \frac{1}{y}$. So $\frac{dv}{dx} = -\frac{1}{y^2} \frac{dy}{dx}$.

The DE is transformed into the following:

$$-\frac{dv}{dx} + \frac{2}{x} v = -(x^2 \cos x)$$

$$\Longleftrightarrow \quad \frac{dv}{dx} - \frac{2}{x} v = x^2 \cos x$$

Integrating Factor: $\mu(x) = e^{\int -\frac{2}{x} dx} = x^{-2}$.

Multiplying both sides by x^{-2}, we get

$$\frac{d}{dx}\left[x^{-2} v\right] = \cos x.$$

Hence $x^{-2} v = \int \cos x \, dx + c = \sin x + c$

Thus $v = x^2 (\sin x + c)$.

Therefore $\boxed{y = \frac{1}{x^2(\sin x + c)}}$

EX2: Find the value of k that makes the following DE: $(y^3 + kxy^4 - 2x)\,dx + (3xy^2 + 20x^2y^3)\,dy = 0$ ② an exact DE; and solve it.

Solution:

Test of exactness: $\frac{\partial}{\partial y}(y^3 + kxy^4 - 2x) = \frac{\partial}{\partial x}(3xy^2 + 20x^2y^3)$

$\Leftrightarrow 3y^2 + 4kxy^3 = 3y^2 + 40xy^3$

$\Leftrightarrow 4kxy^3 = 40xy^3$

We conclude that $k = 10$.

Now, let us solve the DE (for $k=10$).

Let $F(x,y)$ be a potential of the differential form:

$$\begin{cases} \frac{\partial F}{\partial x} = y^3 + 10xy^4 - 2x & ① \\ \frac{\partial F}{\partial y} = 3xy^2 + 20x^2y^3 & ② \end{cases}$$

From ①, we have $F(x,y) = y^3x + 5x^2y^4 - x^2 + g(y)$

Now, using ②, we obtain

$$3y^2x + 20x^2y^3 + g'(y) = 3xy^2 + 20x^2y^3$$

This yields $g'(y) = 0$; that is $g(y)$ is constant.

It follows that $F(x,y) = y^3x + 5x^2y^4 - x^2$ is a potential; the solutions are given implicitly by the relation $\boxed{y^3x + 5x^2y^4 - x^2 = C}$

EX3 : Use an appropriate substitution to transform the following DE :

$$x^5\,dx + (x^2y^3 + x^3y^2)\,dy = 0$$

into a separable DE.

Solution :

The DE may be written as :

$$\frac{dy}{dx} = -\frac{x^5}{x^2y^3 + x^3y^2} = -\frac{x^3}{y^3 + xy^2} = -\frac{(x^3/x^3)}{y^3/x^3 + \frac{xy^2}{x^3}}$$

$$= -\frac{1}{\left(\frac{y}{x}\right)^3 + \left(y/x\right)^2}$$

So the DE is homogeneous.

We use the substitution $v = y/x$ $(y = xv)$.

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

So the DE becomes : $v + x\dfrac{dv}{dx} = -\dfrac{1}{v^3 + v^2}$

$$\Leftrightarrow \quad \frac{dv}{dx} = \left(\frac{1}{x}\right)\left(\frac{-1}{v^2 + v^3} - v\right) = -\left(\frac{1}{x}\right)\left[v + \frac{1}{v^2 + v^3}\right]$$

which is a separable DE.

EX4 : Solve the system below according to the values of the parameter m :

$$\begin{cases} x + 2y + 3z = 4 \\ x + 3y + 3z = 5 \\ 2x + 6y + 6z = m \end{cases}$$

Solution:

The augmented matrix of the system is

$$M = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 6 & m \end{array}\right]$$

$$M \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 5 \\ 0 & 0 & 0 & m-10 \end{array}\right]$$

- If $m \neq 10$, then the system is inconsistent (has no solution).
- If $m = 10$, then by performing the elementary row operation $R_2 - R_1$, we get

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the system is equivalent to $\begin{cases} x + 3z = 2 \\ y = 1 \end{cases}$

It follows that the solutions are given by

$(x, y, z) = (2-t, 1, t)$, where $t \in \mathbb{R}$.

EX5: Find the reduced row echelon form of the matrix

$$M = \begin{bmatrix} 3 & 1 & -1 & 4 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 3 & 5 \end{bmatrix}$$

Solution:

$$M \xrightarrow{R_{1,3}} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 1 & -1 & 4 \\ 2 & 1 & 3 & 5 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 - 3R_1} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -2 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_2} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 5 & 3 \end{bmatrix} \xrightarrow{\frac{1}{5} R_3} \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 1 & 3/5 \end{bmatrix}$$

$$\xrightarrow[R_2 + 4R_3]{R_1 - R_3} \begin{bmatrix} 1 & 0 & 0 & 7/5 \\ 0 & 1 & 0 & 2/5 \\ 0 & 0 & 1 & 3/5 \end{bmatrix} = RREF(M).$$